Swansea University
Mathematics scholarship exam 2023

2 hours 30 minutes
Calculators allowed, but no formula books.

Please attempt all the questions in section A, and then at most four from section B. Explanations of your solutions will form part of the assessment.

Section A

1. What fractions (written in a form like 7/13) have the repeating decimal expansions

$$(a) \quad 0\cdot(17)^{\bullet} \qquad , \qquad (b) \quad 0\cdot 6(12)^{\bullet} \ .$$

2. Determine whether 899 is prime or not. Show your reasoning.

3. For what real values of x is the following inequality true?

$$\frac{x\,(x^2-9)}{x-2} \geq 0$$

4. The number of bacteria in a culture increases by 6% every minute. If there were initially two million bacteria in the culture, how many would there be after 500 seconds?

5. Evaluate the integral

$$\int_0^{\pi/2} \sin(2x)\, dx \ .$$

6. Find the solution x with $1 \leq x \leq 2$ (there is only one in this region) to the following equation correct to within an error of $\frac{1}{16}$:

$$x^3 \ = \ x+1 \ .$$

7. Find the differential with respect to x of e^{x^3}.

8. Find numbers a, b, c, d so that

$$\frac{x-9}{x^2-3x-4} = \frac{a}{x+b} + \frac{c}{x+d} \ .$$

9. A triangle has sides 4, 5 and 5. What is the size of the angle opposite the side of length 4?

10. Given that $x = 3$ is a root of the cubic $x^3 - 2x^2 - 4x + 3$, find the other two roots.

Section B

1. The function $f(x)$ is defined for real $x \neq 0$ by the formula

$$f(x) = 2\,x + \frac{1}{x} \; .$$

(a) For which real values y is there a solution to the equation $f(x) = y$?

(b) If $x \geq 1$, show that $f(x+1) \geq f(x) + 1$.

(c) For which real values x do we have $f(x) > x$?

2. An archer shoots an arrow of mass m at an angle $\alpha > 0$ to the horizontal with initial velocity $\dot{\underline{\boldsymbol{r}}}(0) = v_0 \cos\alpha \underline{\boldsymbol{i}} + v_0 \sin\alpha \underline{\boldsymbol{k}}$ subject to the force of gravity and a drag of $-Km\,\dot{\underline{\boldsymbol{r}}}$. For convenience, denote the position of the arrow by $\underline{\boldsymbol{r}}(t) = x(t)\underline{\boldsymbol{i}} + z(t)\underline{\boldsymbol{k}}$. Assuming that the arrow's initial position is at the origin O (the archer is standing in a conveniently placed trench) find the time at which the arrow is highest above the ground.

3. A sequence of numbers $a_1, a_2, a_3, \ldots$ begins with $a_1 = 2$ and continues according to the rule

$$a_{n+1} \;=\; 2\,a_n - 1 \; .$$

a) Write down the first six numbers in the sequence.

b) Give a formula for a_n in terms of n which gives the correct result for the first six numbers in the sequence.

c) Prove by induction that your formula in part (b) is correct for all $n \geq 1$. [Careful – it is possible that you may have given a formula for (b) which only works for some values of n and not all of them. However the correct formula is not too complicated.]

4. A manufacturer wishes to maximise the area of a rectangular metal plate of width x and length y. However there is a constraint on x and y of the form $2\,x^2 + y^2 = 8$. (One result of this is that x must be in the range $0 \leq x \leq 2$.)

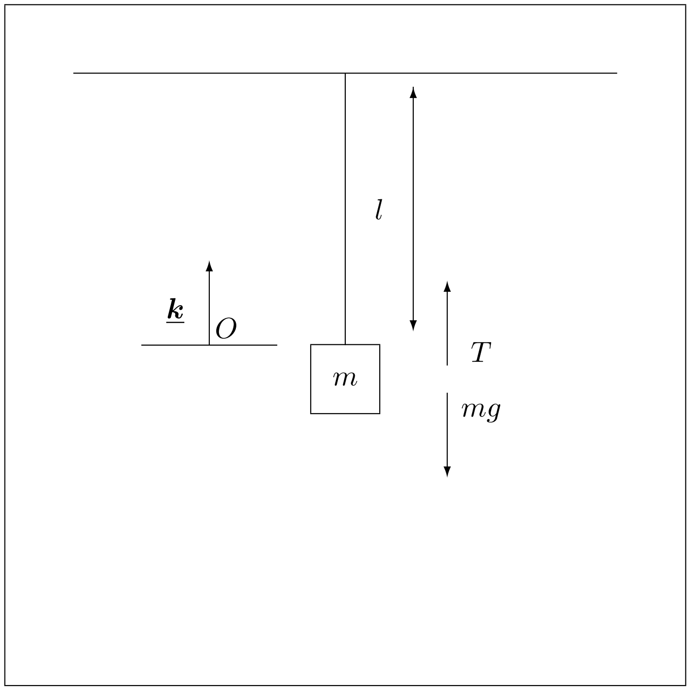
a) Find a formula for the area of the plate involving x only.

b) What value of x maximises the area of the rectangular plate?

c) What is the largest possible area of the rectangular plate?

5. An elastic string of natural length lm with modulus of elasticity λ is attached to a ceiling. A particle of mass m kg is attached to the free (other) end of the elastic string. Let the force of gravity acting on the mass be $-mg\underline{\boldsymbol{k}}$ and consider the origin to be at a distance l below the ceiling.
a) The mass is held at rest such that the elastic string is of length lm and then released, with initial velocity $\underline{\mathbf{0}}$. After what time does the mass return to its initial point?

b) Subsequently, if the particle instantaneously loses 3/4 of its mass each time the elastic string has length l how long does them motion persist?
The diagram below illustrates the setup.



6. A mathematician is travelling by train through the land of knights and knaves to the the town of Smozo. Remember that knights always tell the truth, and that knaves always lie, and that everyone in the land is either a knight or a knave. At a station in a town the mathematician asks a porter 'Is it the case that you are a knight if and only if this is Smozo?'.

a) If the porter's answer is 'yes', is the town Smozo or not, or is there insufficient information to tell?

b) If the porter's answer is 'no', is the town Smozo or not, or is there insufficient information to tell?

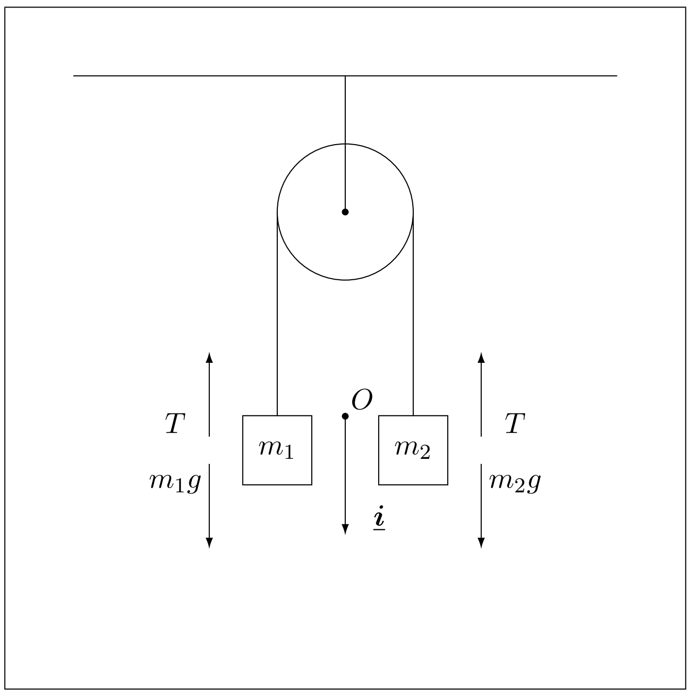
7.

a) Combinatorix the Gaulish chieftain arranges 8 warriors in a line. Given that each warrior can be told apart from the others, how many ways are there to do this?

b) Next Combinatorix chooses 4 of the warriors to attack the nearby Roman fort. How many ways could this choice have been made?

c) The chosen 4 warriors are clothed in stolen Roman uniforms, and because of the helmets, they cannot be told apart. Now Combinatorix again arranges the 8 warriors in a line. How many observable ways can this be done, given that some warriors now look the same?

8. Two particles of masses $m_1 = 5$kg and $m_2 = 2$kg are attached to the ends of a light inextensible elastic string of length $l = 10$m, which hangs over a smooth pivot (pulley) at its midpoint and are held at rest.
i) Calculate the tension T_0 in the elastic string.
ii) At time $t = 0$ the masses are released and move under the influence of gravity and the tension T in the elastic string.
iia) If $x_1(t)$ represents the position of mass m_1 state the equation of motion for mass m_1 and solve it.
iib) Calculate the time at which mass m_2 reaches a position 2.5m from the pivot.
The diagram below illustrates the setup.



9. Two fair 6-sided dice (with faces labelled 1,2,3,4,5,6) are thrown one after the other. What is the probability of the following three events?

a) The numbers on the dice are both even.

b) The number on one of the dice is even, and that on the other is odd. (It does not matter which dice is even, and which is odd.)

c) You cannot see the result of the dice throws, but somebody who can says (truthfully) that the first die throw is even. What is the probability that the numbers on the dice add to 8?