Swansea University
Prifysgol Abertawe

Engineering Analysis 2 : Vectors

P. Rees, O. Kryvchenkova, P.D. Ledger

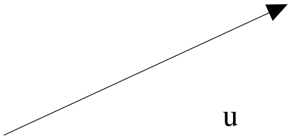
engmaths@swansea.ac.uk

College of Engineering, Swansea University, UK

Outline

Vectors and Scalars

A **vector** is a quantity which has both magnitude as well as direction.



The notations $\boldsymbol{u}$, $\vec{u}$ or $\underline{u}$ are frequently used to distinguish vector quantities, on the slides we will usually use the former.

A **scalar** is a quantity which has magnitude only, we shall use the plain notation ϕ to denote that the quantity is a scalar.

Examples of vector and scalar quanties

Examples of vector and scalar quantities in engineering include:

Vector	Scalar
Acceleration	Temperature
Velocity	Speed
Displacement	Pressure
Force	Density
Electric Field	Permeability
Magnetic Field	viscosity

Plus many more!

The **simplest examples of scalars are numbers.**

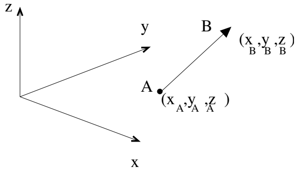
To **make sense of vectors we need a coordinate system** and the simplest vectors have constant components with respect to the chosen system. We do this next.

In **engineering we will also need to deal with scalars and vectors that are not constant** and we will consider this also.

Outline

Cartesian Coordinates

The most common coordinate system is the **Cartesian coordinate system**.



The Cartesian coordinate system is **right handed** in the sense that a rotation from $0x$ to $0y$ advances along $0z$.

Consider two points A and B with coordinates $(x_A, y_A, z_A) = (1, 1, 1)$ and $(x_B, y_B, z_B) = (1.5, 2, 4)$ and the vector $\boldsymbol{a} = \vec{AB}$ given by

$$\boldsymbol{a} = \vec{AB} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \\ 3 \end{pmatrix}$$

The **components** of this vector are $a_x = x_B - x_A = 0.5$, $a_y = y_B - y_A = 1$ and $a_z = z_B - z_A = 3$.

Components of a vector

Two vectors are said to be **equal** if the components of both vectors are the same.

A vector can be multiplied by a scalar to get another vector. If $\boldsymbol{a} = \lambda \boldsymbol{b}$ where λ is some scalar quantity, then if $\lambda > 0$ the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be **parallel** and if $\lambda < 0$ the vectors are said to be **anti–parallel**.

The **magnitude** or length of a vector is

$$|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

A vector with unit magnitude is called the **unit vector**.

A unit vector $\hat{\boldsymbol{a}}$ in the direction of $\boldsymbol{a}$ can be computed as

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Question

If $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ what is $\hat{\boldsymbol{a}}$?

Solution

$|\boldsymbol{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ so $\hat{\boldsymbol{a}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

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Components of a vector

The unit vectors

$$\boldsymbol{i} = \boldsymbol{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{j} = \boldsymbol{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{k} = \boldsymbol{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

point along the x, y and z axis, respectively.

The vector $\boldsymbol{a}$ may be written in terms of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ (or $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$) as

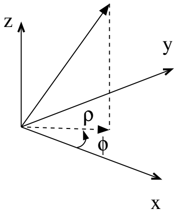
$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} = a_x\boldsymbol{e}_x + a_y\boldsymbol{e}_y + a_z\boldsymbol{e}_z$$

For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with components a_x, a_y, a_z and b_x, b_y, b_z the addition of the two vectors can be written as

$$\begin{aligned} \boldsymbol{a} + \boldsymbol{b} =& (a_x + b_y)\boldsymbol{i} + (a_y + b_y)\boldsymbol{j} + (a_z + b_z)\boldsymbol{k} \\ =& (a_x + b_x)\boldsymbol{e}_x + (a_y + b_y)\boldsymbol{e}_y + (a_z + b_z)\boldsymbol{e}_z \end{aligned}$$

Cylindrical Coordinates - 1

An alternative coordinate systems used when dealing with cylindrical geometries is the cylindrical coordinate system



A point in Cartesian coordinates (x_A, y_A, z_A) can be expressed in cylindrical coordinates as $(\rho_A, \phi_A z_A)$ (and vice versa) using the transformations

$$\rho = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad z = z$$

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z$$

A vector in cylindrical coordinates is expressed in terms of the unit vectors $\boldsymbol{e}_\rho, \boldsymbol{e}_\phi, \boldsymbol{e}_z$ as

$$\boldsymbol{a} = a_\rho \boldsymbol{e}_\rho + a_\phi \boldsymbol{e}_\phi + a_z \boldsymbol{e}_z$$

The cylindrical components a_ρ, and a_ϕ are different to the Cartesian components a_x and a_y, but a_z is the same.

Cylindrical Coordinates - 2

The cylindrical components are related to the Cartesian components as

$$\begin{pmatrix} a_\rho \\ a_\phi \\ a_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = A \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

and the Cartesian components to the Cylindrical components using

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_\rho \\ a_\phi \\ a_z \end{pmatrix} = A^{-1} \begin{pmatrix} a_\rho \\ a_\phi \\ a_z \end{pmatrix}$$

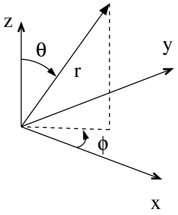
where we have used matrices and A has the special property that $A^{-1} = A^T$ (i.e. it is **orthogonal**).

Note that in general vectors in linear algebra do not have to have length 3 and do not need to be related to a coordinate basis, but when they do they can assist in vector geometry.

Even if $\boldsymbol{a}$ is a constant vector in Cartesian coordinates it is not in cylindrical coordinates. Similarly a constant vector in cylindrical vector is not necessarily constant in Cartesian coordinates. e.g $\boldsymbol{a} = 1\boldsymbol{e}_x + 1\boldsymbol{e}_y + 0\boldsymbol{e}_z = (\cos\phi + \sin\phi)\boldsymbol{e}_\rho + (\cos\phi - \sin\phi)\boldsymbol{e}_\phi + 0\boldsymbol{e}_z$

Spherical Coordinates - 1

An alternative coordinate systems used when dealing with spherical geometries is the **spherical coordinate system**



A point in Cartesian coordinates (x_A, y_A, z_A) can be expressed in spherical coordinates as (r_A, ϕ_A, θ_A) (and vice versa) using the transformations

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad \theta = \cos^{-1}\frac{z}{r}$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

A vector in spherical coordinates is expressed in terms of the unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_\theta$ as

$$\boldsymbol{a} = a_r\boldsymbol{e}_r + a_\phi\boldsymbol{e}_\phi + a_\theta\boldsymbol{e}_\theta$$

The spherical components a_r, a_ϕ and , a_θ are different to the Cartesian coordinates a_x, a_y and a_z, but the component a_ϕ is the same for both cylindrical and spherical.

Spherical Coordinates - 2

The spherical components are related to the cylindrical components as

$$\begin{pmatrix} a_r \\ a_\phi \\ a_\theta \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \end{pmatrix} \begin{pmatrix} a_\rho \\ a_\phi \\ a_z \end{pmatrix} = B \begin{pmatrix} a_\rho \\ a_\phi \\ a_z \end{pmatrix}$$

and the cylindrical components to the spherical components using

$$\begin{pmatrix} a_\rho \\ a_\phi \\ a_z \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & -\sin\theta \end{pmatrix} \begin{pmatrix} a_r \\ a_\phi \\ a_\theta \end{pmatrix} = B^{-1} \begin{pmatrix} a_r \\ a_\phi \\ a_\theta \end{pmatrix}$$

where we have used matrices and again B has the special property that $B^{-1} = B^T$ (i.e. it is **orthogonal**).

Similar operates relate spherical components to Cartesian components.

General Notation

In general we will write

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3$$

where a_1, a_2, a_3 are the components of the vector in the chosen coordinate system (usually Cartesian) and $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the (orthogonal) unit vectors in this system.

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{e}_1 + (a_2 + b_2)\boldsymbol{e}_2 + (a_3 + b_3)\boldsymbol{e}_3$$

The **magnitude** or length of this vector is

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

A unit vector $\hat{\boldsymbol{a}}$ in the direction of $\boldsymbol{a}$ can be computed as

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Example

Question
Determine the vector $\boldsymbol{a} = \vec{OA}$ where O is the origin and A is the point with Cartesian coordinates $(-1, -2, -3)$
Solution

$$\boldsymbol{a} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = -1\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k} = -1\boldsymbol{e}_x - 2\boldsymbol{e}_y - 3\boldsymbol{e}_z$$

Question
Determine the vector $\boldsymbol{a} + \boldsymbol{b}$ where $\boldsymbol{a} = \vec{OA}$, $\boldsymbol{b} = \vec{OB}$ with O the origin, A the point $(-1, -2, -3)$ and B the point $(2, 1, 3)$ in Cartesian coordinates. Subsequently determine $|\boldsymbol{a} + \boldsymbol{b}|$
Solution
$\boldsymbol{a} = -1\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$. Thus

$$\boldsymbol{a} + \boldsymbol{b} = (-1 + 2)\boldsymbol{i} + (-2 + 1)\boldsymbol{j} + (-3 + 3)\boldsymbol{k} = \boldsymbol{i} - \boldsymbol{j}$$

Alternatively this could be written in terms of $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$ instead of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

Also $|\boldsymbol{a} + \boldsymbol{b}| = \sqrt{(1)^2 + (-1)^2 + 0^2} = \sqrt{2}$

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Example

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Question

A vector in cylindrical coordinates is $\boldsymbol{a} = 0\boldsymbol{e}_r + 1\boldsymbol{e}_\phi + \boldsymbol{e}_z$. Write down an expression for this vector in Cartesian coordinates. Where is the magnitude of this vector a maximum?

Solution

In Cartesian coordinates this vector has components

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

and thus

$$\boldsymbol{a} = -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y + 0\boldsymbol{e}_z = -\frac{y}{\sqrt{x^2+y^2}}\boldsymbol{e}_x + \frac{x}{\sqrt{x^2+y^2}}\boldsymbol{e}_y + 0\boldsymbol{e}_z$$

Its magnitude in both cylindrical coordinates and in Cartesian coordinates is the same

$$|\boldsymbol{a}| = \sqrt{0^2+1^2+0^2} = \sqrt{(-\sin\phi)^2+\cos^2\phi+0^2} = 1$$

and is constant everywhere.

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and thus

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Its magnitude in both cylindrical coordinates and in Cartesian coordinates is the same

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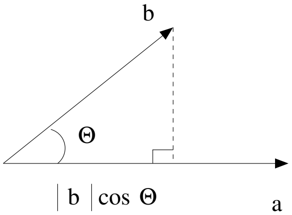
Outline

Dot Products

The dot products between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a **scalar** given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \Theta$$

where Θ is the angle between the two vectors.



Another expression is

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties of Dot Products

- The dot product is commutative $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$;
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** (orthogonal) $\boldsymbol{a} \cdot \boldsymbol{b} = 0$;
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular or $\boldsymbol{a}$ or $\boldsymbol{b}$ or both are zero;
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$;
- $\boldsymbol{a} \cdot \boldsymbol{b}$ is the magnitude of $\boldsymbol{a}$ multiplied by the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.
- The dot product is distributive over addition $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$.

Dot Products in Orthogonal Coordinates

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3) \cdot (b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3)$$

Note that $\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = 1$, $\boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = 1$, $\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1$ and that $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \boldsymbol{e}_1 \cdot \boldsymbol{e}_3 = \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = 0$, for the orthogonal coordinate systems we have previously considered, then

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= a_1b_1\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 + a_2b_2\boldsymbol{e}_2 \cdot \boldsymbol{e}_2 + a_3b_3\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 \\ &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

Example

Question
Given the Cartesian vectors $\boldsymbol{a} = \boldsymbol{e}_x - \boldsymbol{e}_y + 2\boldsymbol{e}_z$, $\boldsymbol{b} = \boldsymbol{e}_x + \boldsymbol{e}_y + 2\boldsymbol{e}_z$ and $\boldsymbol{c} = \boldsymbol{e}_x + \boldsymbol{e}_y$ determine a) $\boldsymbol{a} \cdot \boldsymbol{b}$ and b) $\boldsymbol{b} \cdot (\boldsymbol{a} + 2\boldsymbol{c})$
Solution
a) Direct substitution gives

$$\boldsymbol{a} \cdot \boldsymbol{b} = 1(1) + 1(-1) + 2(2) = 4$$

b) First compute $\boldsymbol{a} + 2\boldsymbol{c}$

$$\boldsymbol{a} + 2\boldsymbol{c} = (1 + 2(1))\boldsymbol{e}_x + (-1 + 2(1))\boldsymbol{e}_y + (2 + 2(0))\boldsymbol{e}_z = 3\boldsymbol{e}_x + \boldsymbol{e}_y + 2\boldsymbol{e}_z$$

Then

$$\boldsymbol{b} \cdot (\boldsymbol{a} + 2\boldsymbol{c}) = 1(3) + 1(1) + 2(2) = 8$$

Example

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Given the Cartesian vectors $\boldsymbol{a} = \boldsymbol{e}_x - \boldsymbol{e}_y + 2\boldsymbol{e}_z$, $\boldsymbol{b} = \boldsymbol{e}_x + \boldsymbol{e}_y + 2\boldsymbol{e}_z$ and $\boldsymbol{c} = \boldsymbol{e}_x + \boldsymbol{e}_y$ determine a) $\boldsymbol{a} \cdot \boldsymbol{b}$ and b) $\boldsymbol{b} \cdot (\boldsymbol{a} + 2\boldsymbol{c})$
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Example

Question
Find the angle between the vectors $\boldsymbol{a} = \boldsymbol{e}_x + 2\boldsymbol{e}_y + 3\boldsymbol{e}_z$, $\boldsymbol{b} = 2\boldsymbol{e}_x + 4\boldsymbol{e}_z$
Solution
By definition $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \Theta$. Here

$$\boldsymbol{a} \cdot \boldsymbol{b} = 1(2) + 2(0) + 3(4) = 14$$

Also

$$|\boldsymbol{a}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14} \qquad |\boldsymbol{b}| = \sqrt{(2)^2 + (0)^2 + (4)^2} = \sqrt{20}$$

Giving the final result

$$14 = \sqrt{14}\sqrt{20} \cos \Theta \qquad \Theta \approx 0.580 rad$$

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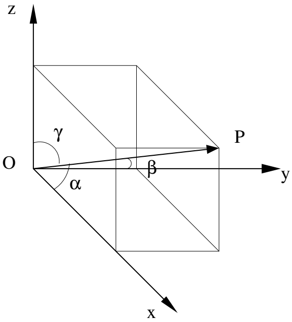
$$|\boldsymbol{a}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14} \qquad |\boldsymbol{b}| = \sqrt{(2)^2 + (0)^2 + (4)^2} = \sqrt{20}$$

Giving the final result

$$14 = \sqrt{14}\sqrt{20} \cos \Theta \qquad \Theta \approx 0.580 rad$$

Direction Cosines

Consider a vector $\boldsymbol{r} = \vec{OP}$ and the angles it makes with the coordinate axis



The unit vector in the direction of $\boldsymbol{r}$ can be expressed in term of its Cartesian components as

$$\begin{aligned}\hat{\boldsymbol{r}} &= l\boldsymbol{e}_x + m\boldsymbol{e}_y + n\boldsymbol{e}_z \\ &= \cos\alpha\boldsymbol{e}_x + \cos\beta\boldsymbol{e}_y + \cos\gamma\boldsymbol{e}_z\end{aligned}$$

Direction Cosines

The direction cosines l, m, n are the cosines of the angles between a vector $\boldsymbol{r}$ and the coordinate axis. By application of the dot product:

$$l = \cos\alpha = \frac{\boldsymbol{r}\cdot\boldsymbol{e}_x}{|\boldsymbol{r}|} = \frac{x_P}{\sqrt{x_P^2 + y_P^2 + z_P^2}}$$

$$m = \cos\beta = \frac{\boldsymbol{r}\cdot\boldsymbol{e}_y}{|\boldsymbol{r}|} = \frac{y_P}{\sqrt{x_P^2 + y_P^2 + z_P^2}}$$

$$n = \cos\gamma = \frac{\boldsymbol{r}\cdot\boldsymbol{e}_z}{|\boldsymbol{r}|} = \frac{z_P}{\sqrt{x_P^2 + y_P^2 + z_P^2}}$$

They also satisfy

$$l^2 + m^2 + n^2 = 1$$

Note that direction cosines are frequently used in Surveying.

Example

Question

If P has Cartesian coordinates $(2, -1, 3)$, find the direction cosines of $\vec{OP}$

Solution

$r = |\vec{OP}| = \sqrt{(2)^2 + (-1)^2 + (3)^2} = \sqrt{14}$ and thus the direction cosines are

$$l = \frac{2}{\sqrt{14}} \qquad m = -\frac{1}{\sqrt{14}} \qquad n = \frac{3}{\sqrt{14}}$$

Example

Question

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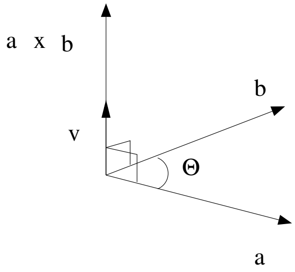
Cross Product

The cross product between two vectors is a **vector** given by $\boldsymbol{a} \times \boldsymbol{b}$.



- $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin \Theta$ where Θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$
- $|\boldsymbol{a} \times \boldsymbol{b}|$ is also the area of parallelogram made by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$;
- The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$
- The cross product is not commutative $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$;
- If the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$;

Thus $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin \theta \boldsymbol{v}$

Cross Product in Orthogonal Coordinates

$$\boldsymbol{a} \times \boldsymbol{b} = (a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3) \times (b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3)$$

To find an explicit expression it is easiest to consider the Cartesian case where $\boldsymbol{e}_1 = \boldsymbol{e}_x$, $\boldsymbol{e}_2 = \boldsymbol{e}_y$, $\boldsymbol{e}_3 = \boldsymbol{e}_z$. But the same results holds for the other orthogonal coordinate systems as well.

Consider $\boldsymbol{e}_1 \times \boldsymbol{e}_2$, since these two vectors have magnitude 1 and are perpendicular $\sin\Theta = 1$ and therefore the magnitude of $\boldsymbol{e}_1 \times \boldsymbol{e}_2$ is 1. The direction of $\boldsymbol{e}_1 \times \boldsymbol{e}_2$ perpendicular to both $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ and so $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$.

In a similar way $\boldsymbol{e}_1 \times \boldsymbol{e}_1 = \boldsymbol{0}$, $\boldsymbol{e}_2 \times \boldsymbol{e}_2 = \boldsymbol{0}$, $\boldsymbol{e}_3 \times \boldsymbol{e}_3 = \boldsymbol{0}$, $\boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1$ and $\boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2$.

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= a_1b_2\boldsymbol{e}_1 \times \boldsymbol{e}_2 + a_1b_3\boldsymbol{e}_1 \times \boldsymbol{e}_3 + a_2b_1\boldsymbol{e}_2 \times \boldsymbol{e}_1 + a_2b_3\boldsymbol{e}_2 \times \boldsymbol{e}_3 + \\ &\qquad a_3b_1\boldsymbol{e}_3 \times \boldsymbol{e}_1 + a_3b_2\boldsymbol{e}_3 \times \boldsymbol{e}_2 \\ &= (a_2b_3 - a_3b_2)\boldsymbol{e}_1 + (a_3b_1 - a_1b_3)\boldsymbol{e}_2 + (a_1b_2 - a_2b_1)\boldsymbol{e}_3,\end{aligned}$$

Cross Product using Determinants

Recall that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The vector product of $\boldsymbol{a} \times \boldsymbol{b}$ with $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3$ and $\boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3$ is equivalent to

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{e}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{e}_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{e}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Scalar Triple Product

Scalar triple product is a **scalar quantity** and is defined as

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$$

Properties:

- The dot and the cross may be interchanged $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$;
- The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ may be permuted cyclically $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})$;

Using the previous results, we have that

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = a_1 b_2 c_3 - a_1 b_3 c_2 + a_2 b_3 c_1 - a_2 b_1 c_3 + a_3 b_1 c_2 - a_3 b_2 c_1$$

Scalar Triple Product with Determinates

An easy way to remember the scalar triple product of $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ is using the determinant

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector Triple Product

The vector triple product is a **vector** quantity and is defined as

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$$

The brackets are important here as $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \neq (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$.

Properties:

- $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$;
- $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = -\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = -(\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a} + (\boldsymbol{c} \cdot \boldsymbol{a})\boldsymbol{b}$.

Example

Question
For the Cartesian vectors $\boldsymbol{a} = 1\boldsymbol{e}_x + 2\boldsymbol{e}_y + 3\boldsymbol{e}_z$, $\boldsymbol{b} = 1\boldsymbol{e}_x + 1\boldsymbol{e}_y + 1\boldsymbol{e}_z$ and $\boldsymbol{c} = 1\boldsymbol{e}_x + 0\boldsymbol{e}_y + 3\boldsymbol{e}_z$. Determine $\boldsymbol{a} \times \boldsymbol{b}$, $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ and $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$

Solution
By direct substitution

$$\boldsymbol{a} \times \boldsymbol{b} = (2(1) - 3(1))\boldsymbol{e}_x + (3(1) - 1(1))\boldsymbol{e}_y + (1(1) - 2(1))\boldsymbol{e}_z = -\boldsymbol{e}_x + 2\boldsymbol{e}_y - \boldsymbol{e}_z$$

(Alternatively the determinate method might be used)
To obtain $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$, first compute $\boldsymbol{b} \times \boldsymbol{c}$ to give

$$\boldsymbol{b} \times \boldsymbol{c} = (1(3) - 1(0))\boldsymbol{e}_x + (1(1) - 1(3))\boldsymbol{e}_y + (1(0) - 1(1))\boldsymbol{e}_z = 3\boldsymbol{e}_x - 2\boldsymbol{e}_y - 1\boldsymbol{e}_z$$

Then

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = 1(3) + 2(-2) + 3(-1) = -4$$

(Alternatively the determinate method might be used)
For $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$, we use the result for $\boldsymbol{b} \times \boldsymbol{c}$ and find

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (2(-1) - 3(-2))\boldsymbol{e}_x + (3(3) - 1(-1))\boldsymbol{e}_y + (1(-2) - 2(3))\boldsymbol{e}_z = 4\boldsymbol{e}_x + 10\boldsymbol{e}_y - 8\boldsymbol{e}_z$$

Example

Question
For the Cartesian vectors $\boldsymbol{a} = 1\boldsymbol{e}_x + 2\boldsymbol{e}_y + 3\boldsymbol{e}_z$, $\boldsymbol{b} = 1\boldsymbol{e}_x + 1\boldsymbol{e}_y + 1\boldsymbol{e}_z$ and $\boldsymbol{c} = 1\boldsymbol{e}_x + 0\boldsymbol{e}_y + 3\boldsymbol{e}_z$. Determine $\boldsymbol{a} \times \boldsymbol{b}$, $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ and $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$
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By direct substitution

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Outline

Physical scalar and vector fields

Recall that examples of vector and scalar quantities in engineering include:

Vector	Scalar
Acceleration	Temperature
Velocity	Speed
Displacement	Pressure
Force	Density
Electric Field	Permeability
Magnetic Field	viscosity

Each of the scalar vector quantities is often not the same at all locations in space, in reality they all vary. In other words, they are examples of **fields**.

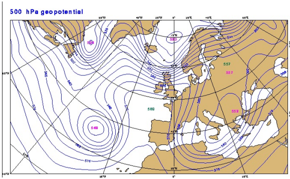
Scalar fields are scalars that take different values at different points in space.

Vector fields are vectors whose components take different values at different points in space.

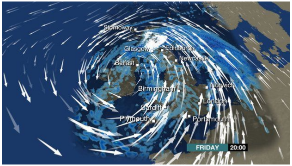
We can choose to work with either Cartesian, cylindrical or spherical coordinates, write the physical vectors in the appropriate components and apply all the aforementioned products as desired.

Visualisation of Fields

You will be familiar with visualising scalar and vector fields even if you don't realise it!



Pressure Contours



Wind Velocity Quiver Plot

Contours have the same value of a scalar field (e.g. Pressure). In quiver plot, the arrows indicate the direction and length of the arrow its magnitude (e.g. Wind velocity).

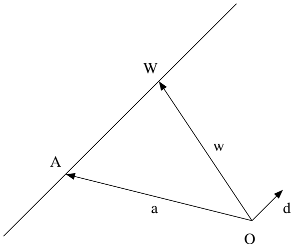
To understand the rates at which fields change we need to extend our ideas of differentiation and we will do this later in EG190. We'll learn about this when considering multivariate functions.

Outline

Vector Equation of a Line

Let $A = (x_a, y_A, z_A)$ be a known point on a line and $\boldsymbol{d} = l\boldsymbol{e}_x + m\boldsymbol{e}_y + n\boldsymbol{e}_z$ a known Cartesian direction vector for the line.

Note that $\boldsymbol{d}$ can be any vector that is parallel to the line.



We want to find an equation for $W = (x, y, z)$, a general point on the line.

Set $\boldsymbol{a} = \vec{OA}$ and $\boldsymbol{w} = \vec{OW}$. The vector $\vec{AW} = \boldsymbol{w} - \boldsymbol{a}$ is parallel to the line and so must be a scalar multiple of $\boldsymbol{d}$. Thus

$$\boldsymbol{w} - \boldsymbol{a} = t\boldsymbol{d}$$

for some t. Hence $\boldsymbol{w} = \boldsymbol{a} + t\boldsymbol{d}$ which is the **vector from of the equation of the line**.

Cartesian Equation of a Line

Now by equating components

$$x = x_A + tl \qquad y = y_A + tm \qquad z = z_A + tn$$

Then, provided that l, m, n are non-zero, the **Cartesian form of the equation of the line** is

$$\frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n} \qquad (= t)$$

Conversely if the equation of the line is known as

$$\frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n} \qquad (= t)$$

then (x_A, y_A, z_A) is a point on the line and $\boldsymbol{d} = l\boldsymbol{e}_x + m\boldsymbol{e}_y + n\boldsymbol{e}_z$ is a direction vector for the line.

Example

Question
Determine the (vector and Cartesian) equation of the line containing the points A with Cartesian coordinates $(-1,-2,3)$ and B with Cartesian coordinates $(1,1,2)$.
Solution
A direction vector for the line is $\boldsymbol{d}=\vec{AB}=2\boldsymbol{e}_x+3\boldsymbol{e}_y-\boldsymbol{e}_z$. Also $\boldsymbol{a}=\vec{OA}=-\boldsymbol{e}_x-2\boldsymbol{e}_y+3\boldsymbol{e}_z$
The equation of the line in vector form is

$$\begin{aligned} w &= \boldsymbol{a}+t\boldsymbol{d}=(-\boldsymbol{e}_x-2\boldsymbol{e}_y+3\boldsymbol{e}_z)+t(2\boldsymbol{e}_x+3\boldsymbol{e}_y-\boldsymbol{e}_z) \\ w &= (-1+2t)\boldsymbol{e}_x+(-2+3t)\boldsymbol{e}_y+(3-t)\boldsymbol{e}_z \end{aligned}$$

The Cartesian form of the equation of the line is

$$\frac{x+1}{2}=\frac{y+2}{3}=\frac{z-3}{-1}$$

Example

Question
Determine the (vector and Cartesian) equation of the line containing the points A with Cartesian coordinates $(-1,-2,3)$ and B with Cartesian coordinates $(1,1,2)$.
Solution
A direction vector for the line is $\boldsymbol{d} = \vec{AB} = 2\boldsymbol{e}_x + 3\boldsymbol{e}_y - \boldsymbol{e}_z$. Also $\boldsymbol{a} = \vec{OA} = -\boldsymbol{e}_x - 2\boldsymbol{e}_y + 3\boldsymbol{e}_z$
The equation of the line in vector form is

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{a} + t\boldsymbol{d} = (-\boldsymbol{e}_x - 2\boldsymbol{e}_y + 3\boldsymbol{e}_z) + t(2\boldsymbol{e}_x + 3\boldsymbol{e}_y - \boldsymbol{e}_z) \\ \boldsymbol{w} &= (-1+2t)\boldsymbol{e}_x + (-2+3t)\boldsymbol{e}_y + (3-t)\boldsymbol{e}_z \end{aligned}$$

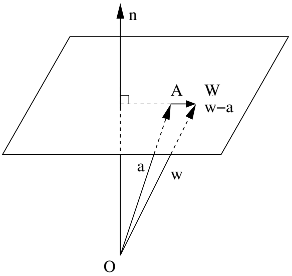
The Cartesian form of the equation of the line is

$$\frac{x+1}{2} = \frac{y+2}{3} = \frac{z-3}{-1}$$

Outline

Vector Equation of a Plane

Let $\boldsymbol{n} = n_x\boldsymbol{e}_x + n_y\boldsymbol{e}_y + n_z\boldsymbol{e}_z$ be a **normal Cartesian vector** which is perpendicular to the plane, $A = (x_A, y_A, z_A)$ be a point on a plane and $W = (x, y, z)$ be a general point on the plane



Set $\boldsymbol{a} = \vec{OA} = x_A\boldsymbol{e}_x + y_A\boldsymbol{e}_y + z_A\boldsymbol{e}_z$ and $\boldsymbol{w} = \vec{OW} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$. Then $\vec{AW} = \boldsymbol{w} - \boldsymbol{a}$ is parallel to the line AW, which lies in the plane, and is perpendicular to $\boldsymbol{n}$. Hence

$$\boldsymbol{n} \cdot (\boldsymbol{w} - \boldsymbol{a}) = 0$$

The **vector form of the equation of a plane** is

$$\boldsymbol{n} \cdot \boldsymbol{w} = \boldsymbol{n} \cdot \boldsymbol{a} = p$$

where p represents the perpendicular distance from, the origin to the plane.

Cartesian Equation of a Plane

Expanding the dot product give the **Cartesian form of the equation of a plane** as

$$n_x x + n_y y + n_z z = p$$

Example

Question

Find the equation of the plane containing the points $A = (1, 1, 1)$, $B = (0, 1, 2)$ and $C = (-1, 1, -1)$.

Solution

First we construct the vectors $\boldsymbol{a} = \vec{OA} = \boldsymbol{e}_x + \boldsymbol{e}_y + \boldsymbol{e}_z$, $\boldsymbol{b} = \vec{OB} = \boldsymbol{e}_y + 2\boldsymbol{e}_z$ and $\boldsymbol{c} = \vec{OC} = -\boldsymbol{e}_x + \boldsymbol{e}_y - \boldsymbol{e}_z$. The vectors $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{e}_x - \boldsymbol{e}_z$ and $\boldsymbol{a} - \boldsymbol{c} = 2\boldsymbol{e}_x + 2\boldsymbol{e}_z$ lie in the plane. The normal vector can be constructed as

$$\boldsymbol{n} = (\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} - \boldsymbol{c}) = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ 1 & 0 & -1 \\ 2 & 0 & 2 \end{vmatrix} = -4\boldsymbol{e}_y$$

The vector form of the equation of the plane is

$$\boldsymbol{w} \cdot (-4\boldsymbol{e}_y) = (x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z) \cdot (-4\boldsymbol{e}_y)$$

which in Cartesian form is

$$0x - 4y + 0z = 1(0) + 1(-4) + 1(0)$$

or simply $y = 1$

Example

Question

Find the equation of the plane containing the points $A = (1, 1, 1)$, $B = (0, 1, 2)$ and $C = (-1, 1, -1)$.

Solution

First we construct the vectors $\boldsymbol{a} = \vec{OA} = \boldsymbol{e}_x + \boldsymbol{e}_y + \boldsymbol{e}_z$, $\boldsymbol{b} = \vec{OB} = \boldsymbol{e}_y + 2\boldsymbol{e}_z$ and $\boldsymbol{c} = \vec{OC} = -\boldsymbol{e}_x + \boldsymbol{e}_y - \boldsymbol{e}_z$. The vectors $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{e}_x - \boldsymbol{e}_z$ and $\boldsymbol{a} - \boldsymbol{c} = 2\boldsymbol{e}_x + 2\boldsymbol{e}_z$ lie in the plane. The normal vector can be constructed as

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or simply $y = 1$