Swansea University
Mathematics scholarship exam 2017

2 hours 30 minutes
Calculators allowed, but no formula books.

Please attempt all the questions in section A, and then at most four from section B. Explanations of your solutions will form part of the assessment.

Section A

1. Write the repeating decimal expansion $3{\cdot}(87)^{\bullet}$ as a fraction with integer numerator and denominator (e.g. something like 57/13). Write the fraction 5/13 as a repeating decimal expansion.

2. Determine whether 667 is prime or not. Show your reasoning.

3. Solve the following simultaneous linear equations for x and y:

$$4x + 6y = 10\ ,\ 3x + 4y = 11\ .$$

4. The number of bacteria in a culture increases by 6% every minute. If there were initially two million bacteria in the culture, how many would there be after 200 seconds?

5. Evaluate the integral

$$\int_0^2 \frac{1}{x+2}\, \mathrm{d}x\ .$$

6. There is one real number x which satisfies $x = 1 - x^3/2$ and $0 \le x \le 1$. Calculate x to within an error of $\frac{1}{16}$.

7. Find the differential with respect to x of

$$x^4 + 6\,x^2 + x\ .$$

8. Find the differential with respect to x of

$$\mathrm{e}^{x^2}\ .$$

9. What is the area of an isosceles triangle with sides 6, 10 and 10?

10. Given that $x = 1$ is a root of the cubic $x^3 + 5\,x^2 - 4\,x - 2$, find the other two roots.

Section B

1. This question is about numbers of factors of integers – for example the integer 6 has four factors, which are 1,2,3,6.

a) How many factors has the number 15?

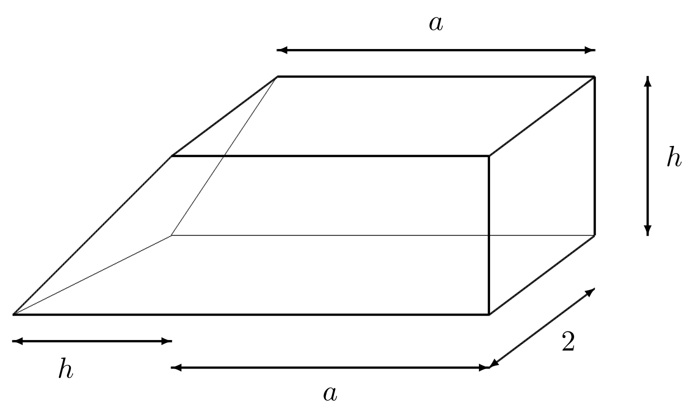
b) How many factors has the number pq, where p and q are different prime numbers? (The answer will not depend on the value of p or q.)

c) How many factors has the number $p^2 q$, where p and q are different prime numbers?

d) How many factors has the number $p^n q^m$, where p and q are different prime numbers and n, m are positive integers?

e) What is the smallest positive integer which has exactly 9 factors.

2. An approximation to the shape of a van is as follows. The side is made by taking a rectangle of height h metres and length a metres and adding an isosceles right angled triangle of sides h, h and $h\sqrt{2}$ metres. The whole van has the same cross section as the side, and has width 2 metres. The sides are vertical, giving the shape pictured below, where the thinner lines are on the far side of the van as we view it:



a) Find the surface area of one of the sides of the van in terms of h and a.
b) Find the total surface area of the van in terms of h and a (add all the surfaces, including the top and bottom).
c) Find the volume of the van in terms of h and a.

The manufacturer decides that the volume of the van should be 16 square metres. Given this, answer the following:
d) Find a formula for a in terms of h, and use it to give a formula for the total surface area of the van just in terms of h.
e) What height h gives the minimum total surface area that the van can have, given the specified volume?

3. A particle of mass m travels in a straight line along a horizontal surface. It is acted on by a resisting force which slows down its motion. Let $x(t)$ denote the distance travelled by the particle up to time t. Given that $x(t)$ is an increasing function of t and that there are positive constants α and β such that $x(t)$ satisfies the equation

$$t = \alpha x + \beta x^2$$

show that the magnitude of the resisting force is proportional to the cube of the speed of the particle.

4.

a) State the principle of induction (any version will do).

b) The following sum is given by a formula for $n \geq 0$, for some constants A and B:

$$\sum_{m=0}^{n} (6\,m + 3) \;=\; 3\,n^2 + A\,n + B\ .$$

Find the constants A and B.

c) Prove by induction that the formula, with your given values of A and B, gives the correct value for the sum for all $n \geq 0$.

5. A castle stands in a large flat empty field. An attacker stands 220m away from the castle with a cannon which launches cannonballs of mass m with a speed 50m/s. His cannon is fixed to fire balls towards the castle at an angle of 30° to the horizontal. A defender stands 100m from the attacker between the attacker and the castle. He has an identical cannon, except that his is fixed to fire towards the attacker at an angle of 60° to the horizontal. The attacker fires his cannon at the castle. The defender waits a time T before firing his cannon with the aim of knocking the attackers cannonball out of the sky. Show that the delay T satisfies,

$$gT^2 + 4\left(1 + \sqrt{3}\right)(25 - g)T - 200 = 0,$$

where g is the acceleration due to gravity.

6. Inspector Code of the Swansea police is investigating a robbery. The suspects are A, B, C and D, and it is known that nobody else could have been involved. The following facts have been established:

i) At least two people committed the robbery.
ii) If A was involved, then exactly three people did the robbery.
iii) If C was involved, then also D is guilty.
iv) If exactly two people were involved, then one of them was B.

Are the following statements definitely true, definitely false, or is it not known whether they are true or false?

(a) All four suspects are guilty.

(b) If exactly two people were involved, then D is guilty.

(c) If exactly three people were involved, then A is guilty.

(d) D is guilty.

7. In this question we consider how many different words can be made out of collections of letters. The word does not need to be in any recognisable language, so for example *adefgbc* and *gbceadf* would be counted in part (a).

a) How many different words can be made out of the letters a, b, c, d, e, f, g? Each letter is to be used exactly once.

b) How many different words can be made out of the letters a, b, c, where each word contains exactly one a, two bs and one c? Thus *abcb* is allowed, but *aabc* is not. [Hint: you could label the two bs as b_1 and b_2, then count the rearrangements, and then remove the labels, which would divide the number of rearrangements by a number.]

c) How many different words can be made out of the letters a, b, c, where each word contains exactly one a, two bs and three cs? Thus *cabbcc* is allowed, but *cabcab* is not.

d) How many different words can be made out of the letters a, b, c, where each word contains exactly one a, two bs and three cs (so far this is the same as the last part), but where we add the restriction that each b must be immediately followed by a c? Thus *abccbc* is allowed, but *abbccc* is not.

8. A stone of mass M is dropped into a well of depth D. Given that the impact of the stone on the bottom of the well is heard T seconds after the stone is dropped, deduce that,

$$D^2 - 2DU\left(T + \frac{U}{g}\right) + T^2 = 0$$

where U is the speed of sound and g is the acceleration due to gravity. Given that $U = 332$ m/s, $g = 9.8$ m/s^2 and it took $T = 1.5$ s to hear the rock hit the bottom, find the depth of the well. How big an error would you have had if you had assumed that sound travelled instantaneously (i.e. $U = \infty$)?

9. In a football league consisting of 4 teams, it is well known that Deeton has the following chances to win against the other teams:
Anystown $\frac{2}{3}$,
Beesville $\frac{1}{2}$,
Ceachester $\frac{1}{3}$.

a) One year Deeton has to beat all the other sides in three games, one against each of the other teams. What is the probability that Deeton can win all three games?

b) The next year Deeton has to win at least one of three matches, one against each of the other teams. What is the probability that Deeton can win at least one of the three games?

c) The year after Deeton has to win one match against one of the other teams, where the opponent is chosen at random (probability $\frac{1}{3}$ that it could be any of the other teams). What is the probability that Deeton can win the game?

d) The year after that Deeton has to win two matches against two of the other teams (one match against each), where the side they do **not** play is chosen at random (probability $\frac{1}{3}$ that it could be any of the other teams). What is the probability that Deeton can win both the games?

10. A curve is given as the locus of points $P = (x, y)$ on the xy-plane such that the distance from P to $F_1 = (1, 0)$ plus the distance from P to $F_2 = (-1, 0)$ is equal to 3, i.e.

$$|PF_1| + |PF_2| = 3.$$

a) Give an equation for the sum of the distances in terms of (x, y). (This should involve square roots.)

b) Show that the equation of the curve is of the form

$$a\,x^2 + b\,y^2 = 1$$

where a and b are constants which you should determine. You will have to use the result of (a) and then use a square operation twice to get rid of all the square roots.