Swansea University
Mathematics scholarship exam 2016

2 hours 30 minutes
Calculators allowed, but no formula books.

Please attempt all the questions in section A, and then at most four from section B. Explanations of your solutions will form part of the assessment.

Section A

1. Write the repeating decimal expansion $2\cdot(56)^\bullet$ as a fraction with integer numerator and denominator (e.g. something like 57/13). Write the fraction 4/37 as a repeating decimal expansion.

2. Determine whether 659 is prime or not. Show your reasoning.

3. Find the product of matrices

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} .$$

4. The number of bacteria in a culture increases by 7% every minute. If there were initially two million bacteria in the culture, how many would there be after 1000 seconds?

5. Evaluate the integral

$$\int_0^2 \frac{2\,x}{x^2+7}\,\mathrm{d}x .$$

6. There is one real number x which satisfies $\tan(x) + 1 = 5x$ and $0 \le x \le 1$. Calculate x to within an error of $\frac{1}{16}$. (Note that x must be in radians, not degrees, for this calculation.)

7. Find the differential with respect to x of

$$(x^2+1)^{101} .$$

8. Find the differential with respect to x of

$$\cos(x^2) .$$

9. A triangle with vertices ABC has the angle at A being 40°, the angle at B being 70° and the length of the side BC being 10. What is the length of the side AC?

10. Given that $x = 2$ is a root of the cubic $x^3 - x^2 - 3x + 2$, find the other two roots.

Section B

1. A can of beans consists of sheet metal in the shape of a right circular cylinder (i.e. the usual shape for a cylindrical can). Given that the radius of the circular base is r and the height of the can is h,

a) find the surface area of the can,

b) find the volume of the can.

We are given a total area of 2π square metres of sheet metal to make a large bean can.

c) Find a formula for h in terms of r.

d) What is the maximum volume can that we could make from this sheet metal?

2. The prime numbers $3, 5, 7$ form a triple separated by 2 each time, i.e. are of the form $p, p+2, p+4$. Explain why it is impossible to find another triple of primes of the form $p, p+2, p+4$.

3. Newton's coefficient of restitution 'e' is a measure of how much kinetic energy will be lost in a collision between two objects. It is defined by,

$$e = \frac{\text{Speed of separation after the collision}}{\text{Speed of approach before the collision}}.$$

Suppose that a ball is dropped from rest from a height h above the ground, and that the coefficient of restitution between the ball and ground is given by some number e where $0 < e < 1$. Deduce that the total distance moved by the ball in the subsequent motion is given by,

$$h\frac{(1+e^2)}{(1-e^2)}.$$

4. The sides of a right angled triangle obey the equation $a^2 + b^2 = c^2$, where c is the hypotenuse. Show that there are infinitely many such right angled triangles where the sides are all integers, and where $c = a + 2$. Can you give a formula covering all these cases?

5. A box of mass $3m$ sits on a shelf which is at a height $2h$ above the floor. The box is attached to the end of a light inextensible string which passes over a smooth pulley at the edge of the shelf. At the other end of the string vertically beneath the pulley hangs a ball of mass m. Initially the box is a distance $2h$ from the edge of the shelf and the ball is a distance h below the pulley. The coefficient of friction between the shelf and the box is μ. Suppose that the system is released from rest. Show that the system will begin to move provided $\mu < 1/3$, and in this case show that when the ball hits the floor the box will be moving with a speed,

$$\sqrt{\frac{1}{2}gh(1-3\mu)}.$$

6. An island is populated entirely by Knights and Knaves, where Knights always tell the truth, and Knaves always lie. On the island you meet a group of three people A, B and C. Then A and B make the following statements:

A says 'At least two of us are Knaves'.

B says 'Both A and C are Knaves'.

a) Is A a Knight or a Knave, or is there insufficient information to tell?

b) Is B a Knight or a Knave, or is there insufficient information to tell?

Sometime later you meet two more inhabitants X and Y.

Y says 'If I am a Knight then X is a Knave'.

c) Is X a Knight or a Knave, or is there insufficient information to tell?

Give reasons for your answers.

7. On the occasion of a royal tournament, there was an entry of $2n$ knights, where n is a positive integer.

a) The first thing they all had to do was to ride past the king in single file. How many possible orders were there for the knights to ride past in?

b) Next the heralds organised the knights into pairs for jousting. Each pair was labelled with the time at which its joust was to take place. In how many possible ways could they have done this pairing? (Note that the order within each pair does not matter.)

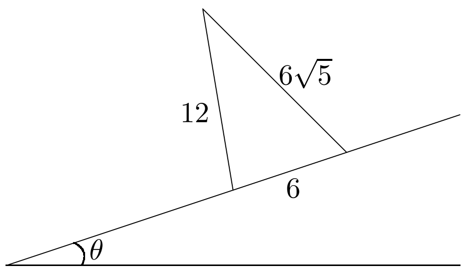
c) As each contest was completed, the heralds announced the result to the crowd. Given that we now know who was in each pair, how many possible results could there be from the whole days competition of n jousts (one for each pair). Note that a draw was not an acceptable outcome.

Now n of the knights (who before had all been distinguishable because they had different shields) are given identical armour and shields, so that they can no longer be told apart.

d) In how many ways could the n knights being given identical armour and shields be chosen from the $2n$ knights?

e) Finally the king chooses n of the $2n$ knights to go on a quest (the order of choice does not matter, only whether they are chosen or not). In how many distinguishable ways could the king do this? Remember that n of the knights cannot be told apart. [You may leave this answer as a sum, rather than looking for a single expression.]

8. Consider a uniform solid triangular prism. The prism has two triangular faces with edges of length $6, 12, 6\sqrt{5}$ and three rectangular faces with dimensions $6 \times h$, $12 \times h$ and $6\sqrt{5} \times h$ where h denotes the length of the prism. The prism is placed on a slope which is at angle of θ radians to the horizontal. The prism rests on its $6 \times h$ rectangular face with its triangular cross section parallel to the slope. The $6\sqrt{5} \times h$ face faces up the slope, whilst the $12 \times h$ face faces down the slope as shown on the figure.



Assuming that the coefficient of friction is such that the prism does not slide down the slope, show that the largest value of θ for which the prism will not topple over satisfies,

$$\sin\theta = \frac{1}{\sqrt{5}}.$$

9. Four fair coins are thrown one after the other. What is the probability of the following three events?

a) There are exactly 3 heads.
b) There are strictly more heads than tails.
c) The first two coins are heads and the total number of heads is three.

Now four fair coins are thrown one after the other as before, but if the fourth coin is a head, and only in that case, a fifth coin is tossed. For this new game;

d) What is the probability that there are exactly 3 heads?
e) What is the probability that the last coin tossed is a tail?

10. A curve is given as the locus of points (x, y) on the xy-plane which are the same distance from the point $F = (0, 8)$ as from the x axis. [Remember that the distance from a point to a line is measured perpendicular to the line.]

a) Give the equation of the curve in the form where y is given as a formula in x.

b) Give the equation of the line tangent to the curve through a point $P = (x_0, y_0)$ on the curve.

c) Give the equation of the line normal to the curve through a point $P = (x_0, y_0)$ on the curve.