Swansea University
Prifysgol Abertawe

Engineering Analysis 1 : Number Systems

Dr. Paul D. Ledger, Dr Rob Daniels, Dr Igor Sazonov

engmaths@swansea.ac.uk

College of Engineering, Swansea University, UK

Outline

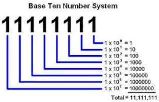
Different ways of Expressing Numbers

We haven't always used the Hindu-Arabic number system : $0, 1, 2, 3, \cdots$ and for many important applications we use other number systems:

Babylonian number system < 1900BC

Time is still measured in base 60

Digitial signals use base 2...

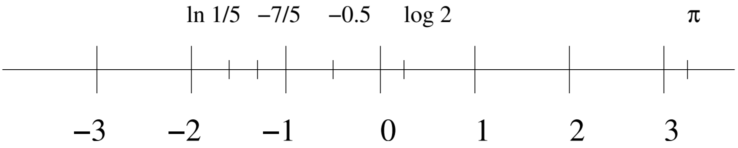


Base 10

For details of how to convert between number systems see the recommended text books.

Types of Numbers

- **Natural numbers** refers to the set $\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\}$
- **Integer Numbers** are the set $\mathbb{Z} = \{\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$. Integers can be represented as equally spaced points on the number line
- The set of all points (ie not only those represented by integers) represents the **real numbers**, it is denoted by the symbol $\mathbb{R}$.
  - A real number which can be written as the ratio of two integers is called a **rational number** eg $\frac{1}{3}$ and $-\frac{7}{5}$.
  - Other numbers like π and $\sqrt{2}$ which can not be expressed in this way are called **irrational numbers**



Powering and Order of Precedence

Recall that $a \times a$ is written as a^2 and $a \times a \times a$ as a^3, In general the product of n a's is written as a^n. We call n the **index** or **exponent**. Note the simple rules

- $a^n \times a^m = a^{n+m}$
- $a^n \div a^m = a^{n-m}$
- $(a^n)^m = a^{nm}$

As a consequence of these rules we have $(a^{1/n})^n = a$, $a^0 = 1$ and $a^{1/n} = \sqrt[n]{a}$.
An order of precedence is observed

- the operation $(\cdot)^r$ is performed first
- then $\times$ or $\div$
- then $+$ or $-$

When two operators of equal precedence are adjacent to each other, the **left hand rule** is applied. The **precedence** is **over written by brackets.**

Comparison Symbols

We make compare quantities using the **comparison symbols** $>$ “greater than” and $<$ “less than”. Other comparison symbols that we commonly use are $=$ “equal to”, $\neq$ “not equal to”, $\geq$ “greater or equal to” and $\leq$ “less than or equal to”

The common rules for comparison symbols are

- **Rule 1** $(a < b)$ and $(c < d)$ implies $a + c < b + d$
- **Rule 2** $(a < b)$ and $(c > d)$ implies $a - c < b - d$
- **Rule 3** $(a < b)$ and $(b < c)$ implies $a < c$ and $a < b$ implies $a + c < b + c$
- **Rule 4** $(a < b)$ and $(c > 0)$ implies $ac < bc$
- **Rule 5** $(a < b)$ and $(c < 0)$ implies $ac > bc$
- **Rule 6** $(a < b)$ and $(ab > 0)$ implies $\frac{1}{a} > \frac{1}{b}$

Example

Question
Find the value of x for which

$$\frac{1}{2-x} < 1$$

Solution
Note that we have **2 solutions**. When $2-x>0$, that is when $x<2$, we can use Rule 4, where we multiply by $2-x$ to give

$$1 < (2-x)$$

which reduces to $x<1$. Which means that (1) is satisfied when $x<2$ and $x<1$, that is when $x<1$.

We now consider $2-x<0$, that is when $x>2$. we use Rule 5, where we multiply by $2-x$ to give

$$1 > (2-x)$$

so that (1) is satisfied when $x>1$ and $x>2$, that is when $x>2$. Thus the inequality is satisfied for values of x such that $x>2$ or $x<1$.

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The Modulus and Open and Closed Sets

The size of a real number x is called its **modulus** and is denoted by $|x|$

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

$|x|$ is the distance on the number line from the point representing 0. Similarly $|x - a|$ is the distance of the point representing x on the number line from that representing a.

The **set of numbers**

$$(a, b) = \{x : a < x < b, x \text{ in } \mathbb{R}\}$$

defines the **open interval**, which does **not** contain the end points. Here we use the notation $\{x : P\}$ which means that each value x of the set has the property P.

An interval that includes the two end points is called a **closed interval** denoted by

$$[a, b] = \{x : a \leq x \leq b, x \text{ in } \mathbb{R}\}$$

Example

Question

Express the set $\{x : |x + 2| < 5, x \text{ in } \mathbb{R}\}$ as an interval.

Solution

First we note that we can write $|x + 2| < 5$ as $|x - (-2)| < 5$

This means that the distance of the point x on the number line from the point representing -2 is less than 5 units.

Thus $-5 < x + 2 < 5$ which means that $-7 < x < 3$ and the set of numbers which satisfy the inequality is the open interval $(-7, 3)$

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Outline

Quadratic Equations

A quadratic equation is of the form

$$y = ax^2 + bx + c$$

A common application requires determining x such that $y = ax^2 + bx + c = 0$ (ie finding the **roots**.

Solution methods include

- Factorisation
- Plotting the graph of $y = ax^2 + bx + c$ and looking for the points where the graph of the function passes through the x axis.
- Completing the square
- Using the standard formula

All are **equivalent**.

Completing the square

Completing square means that

$$y = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

We can show that these are equivalent

$$\begin{aligned} a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} &= a\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) + c - \frac{b^2}{4a} \\ &= a\left(x^2 + \frac{b^2}{4a^2} + \frac{2xb}{2a}\right) + c - \frac{b^2}{4a} \\ &= ax^2 + bx + c \end{aligned}$$

We can use completing the square to find the solution to a quadratic equation $y = 0$, although it is best expressed in terms of the general formula

General for roots of a quadratic equation

Given $ax^2 + bx + c = 0$ and using completing the square

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$$

Rearranging

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This equation gives rise to the following implications

- For $b^2 > 4ac$ we have two real roots
- For $b^2 = 4ac$ we have one repeated root
- For $b^2 < 4ac$ we have no real roots.

Example

Question

Determine the roots of the quadratic equation

$$x^2 + 3x + 2 = 0$$

Solution

To find the roots we simply apply equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 1$, $b = 3$ and $c = 2$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)}}{2} = \frac{-3 \pm 1}{2} = -1 \text{ or } -2$$

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The Number j

Recall that $a^2 \geq 0$ for any **real** number a and that square root of a negative **real** number is not defined as a real number.

In this part of the course we shall introduce a new set of numbers that allow us to make sense of numbers such as $\sqrt{-9}$.

In particular we introduce a new number, j, for which

$$j^2 = -1 \qquad \text{so that} \qquad j = \sqrt{-1}$$

j is not real and is instead an **imaginary** number. The symbol i is sometimes used in place of j.

We can now make sense of $\sqrt{9} = \sqrt{-1}\sqrt{9} = j3$

The Complex Number $z = s + jt$

Recall that the general roots of $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This result gives rise to the following implications

- For $b^2 > 4ac$ we have two real roots
- For $b^2 = 4ac$ we have one repeated root
- For $b^2 < 4ac$ we have no real roots.

We can now make sense of the case $b^2 < 4ac$ in terms of j. We will see in the next slide that the result are two **complex numbers** each expressed in **Cartesian form**

$$z = s + jt$$

where Re$(z) = s$ is called the real part of z and Im$(z) = t$ is called the imaginary part of z. The set of all complex numbers is $\mathbb{C}$.

Example

Question

Determine the roots of the quadratic equation

$$x^2 - 6x + 10 = 0$$

Solution

To find the roots we simply apply the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 1$, $b = -6$ and $c = 10$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(10)}}{2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm j = 3 + j \text{ or } 3 - j$$

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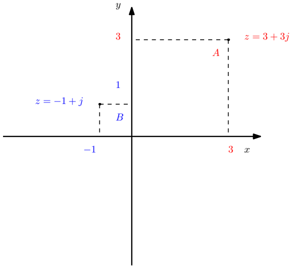
Outline

Agrand Diagram

Complex numbers in Cartesian form $z = x + jy$ can be displayed graphically on an **Argand diagram**. This resembles a graph with each individual complex number being a point with coordinates (x, y).

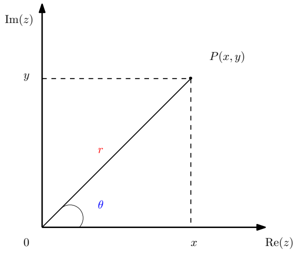
The x-axis corresponds to Re(z) and the y-axis to Im(z).



The point A with coordinates $(3, 3)$ corresponds to $z = 3 + j3$ and the point B with coordinates $(-1, 1)$ corresponds to $z = -1 + j$.

Agrand Diagram

The Argand diagram also makes it possible to introduce an alternative representation of a complex number.



The length of the line OP is the **modulus** of the complex number

$$r = |z| = \sqrt{x^2 + y^2}$$

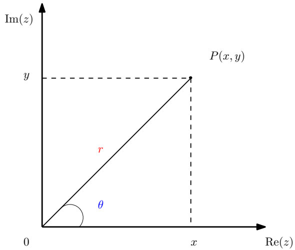
The angle that the line OP makes with the positive x real axis is the **argument** of the complex number

$$\theta = \arg z = \tan^{-1}(y/x)$$

Note that the polar coordinates (r, θ) and $(r, \theta + 2\pi)$ represent the same point on the Argand diagram.

Outline

Polar Form of a Complex Number



The Argand diagram makes it clear that

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

It therefore follows that the complex number $z = x + jy$ can be expressed in the form

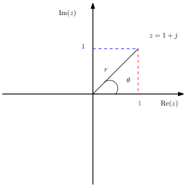
$$z = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta)$$

This is called the **polar form** of the complex number and is frequently written as $r\angle\theta$

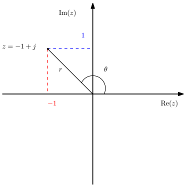
$$z = r\angle\theta = r(\cos\theta + j\sin\theta)$$

Computing the Argument

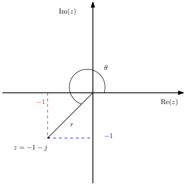
Great **care** is required when applying $\theta = \arg z = \tan^{-1}(y/x)$ on a calculator!



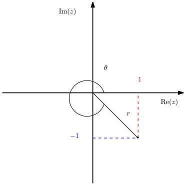
$$z = 1 + j = 1 + 1j$$
$$|z| = \sqrt{2},\ \theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$$



$$z = -1 + j = -1 + 1j$$
$$|z| = \sqrt{2},\ \theta = \pi + \tan^{-1}\frac{1}{-1} = \frac{3\pi}{4}$$



$$z = -1 - j = -1 + (-1)j$$
$$|z| = \sqrt{2},\ \theta = \pi + \tan^{-1}\frac{-1}{-1} = \frac{5\pi}{4}, (-\frac{3\pi}{4})$$



$$z = 1 - j = 1 + (-1)j$$
$$|z| = \sqrt{2},\ \theta = 2\pi + \tan^{-1}\frac{-1}{1} = \frac{7\pi}{4}, (-\frac{\pi}{4})$$

Example

Question
For $z_1 = 2 + j3$ and $z_2 = 3 - j2$ determine their polar form
Solution
For z_1 we have

$$\begin{aligned} r_1 = |z_1| &= \sqrt{2^2 + 3^2} = \sqrt{13} \\ \theta_1 = \arg z_1 &= \tan^{-1}\frac{3}{2} = 0.982(3dp) \end{aligned}$$

Thus $z_1 = 2 + j3 = \sqrt{13}(\cos 0.982 + j\sin 0.982)$. For z_2 we have

$$\begin{aligned} r_2 = |z_2| &= \sqrt{3^2 + (-2)^2} = \sqrt{13} \\ \theta_2 = \arg z_2 &= -\tan^{-1}\frac{2}{3} = -0.588(3dp) \end{aligned}$$

This means that
$z_2 = 3 - j2 = \sqrt{13}(\cos(-0.588) + j\sin(-0.588)) = \sqrt{13}(\cos 0.588 - j\sin 0.588)$

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