Swansea University
Prifysgol Abertawe

Engineering Analysis 2 : Multivariate functions

P. Rees, O. Kryvchenkova and P.D. Ledger,

engmaths@swansea.ac.uk

College of Engineering, Swansea University, UK

Outline

Introduction

Recall from EG189: analysis of a function of single variable:

- Differentiation

$$\frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x}$$

  Expresses the rate of change of a function f with respect to x.

- Integration

$$\int f(x)\, \mathrm{d}\, x$$

  Reverses the operation of differentiation and can used to work out areas under curves, and as we have just seen to solve ODEs.

We've seen in **vectors** we often need to deal with **physical fields**, that depend on more than one variable.

We may be interested in rates of change to each coordinate (e.g. x, y, z), time t, but sometimes also other physical fields as well.

Introduction (Continue ...)

Example 1: the area A of a rectangular plate of width x and breadth y can be calculated

$$A = xy$$

The variables x and y are independent of each other.

In this case, the dependent variable A is a function of the two independent variables x and y as

$$A = f(x, y)$$

or

$$A \equiv A(x, y)$$

Introduction (Continue ...)

Example 2: the volume of a rectangular plate is given by

$$V = xyz$$

where the thickness of the plate is in z-direction.

In this case:

- V — is the dependent variable
- x, y, and z — are the independent variables
- We write $V = f(x, y, z)$ or $V \equiv V(x, y, z)$.

Visualisation of Functions of Two and Three Variables

In general a function f can be a function of n independent variables. But, here we restrict ourselves to functions of two or three independent variables.

If $n = 2$ then we can write

$$f(x, y)$$

This is a function of two independent variables x and y.
How to visualise such a function:

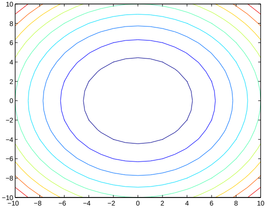
1. Use **level curves** which trace in the x, y domain on which the function $f(x, y)$ has a constant value.
2. Plot the points corresponding to (x, y, z) where we choose $z = f(x, y)$ in a rectangular coordinate system. ie the values of z are chosen to be the values of the function for a given x, y (rather than the z coordintates) and are shown as a surface.

Visualisation of Functions of Two and Three Variables

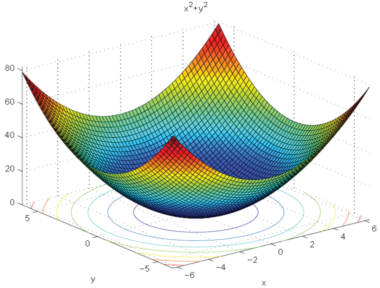
Example: We wish to visualise the function

$$f(x,y) = x^2 + y^2$$

Solution: By using MATLAB, we can make a level surface plot and/or a surface plot of this functions as:



(a) The level surface plot



(b) The surface plot

Outline

Partial Differentiation

1st Semester: the derivative of a function $f(x)$ measures the slope of the tangent to the graph of the function.

If we have a function $f(x,y)$ of two variables, slope has no sense because $z = f(x,y)$ defines a surfaces in 3D.

- Simplest surface $f(x,y) = 0$ is flat in both the x and y directions.

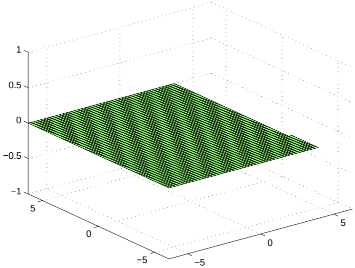


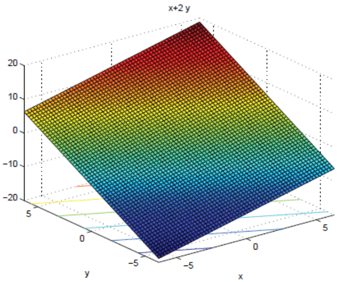
Figure: Visualisation surfaces for $f(x,y) = 0$

Partial Differentiation

Consider $f(x, y) = x + 2y$

i if we move along a line of fixed y and increasing x the slope is equal to 1.

ii if we move along a line for which x is fixed and y is increasing then we find that the slope is equal to 2.



Visualisation of surface for $f(x, y) = x + 2y$

Partial Differentiation

For a general surface the slope will be different depending on which direction we move in.

To find out rates of change for functions of more than 1 variable we need a new kind of derivative —called a **partial derivative**.

The partial derivative of $f(x, y)$ with respect to x is defined as

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

i.e. we differentiate: $f(x, y)$ with respect to x while keeping y constant (fixed).
The partial derivative of $f(x, y)$ with respect to x is the same as measuring the slope in the x direction.

$$\frac{\partial f}{\partial x} \quad \text{or} \quad \partial f / \partial x \quad \text{or} \quad f_x$$

Partial Differentiation

In writing, care must be taken to distinguish between:

$$\frac{df}{dx}, \qquad \frac{\Delta f}{\Delta x} \qquad \text{and} \qquad \frac{\partial f}{\partial x}$$

Similarly, the partial derivative of $f(x, y)$ with respect to y is

$$\frac{\partial f}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Differentiating $f(x, y)$ with respect to y by keeping x constant. This partial derivative is the same as measuring the slope in the y direction.

Partial Differentiation

Directional derivative

If we know $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, we know the surface for any direction.

If we define α — an angle to the x axis then the slope is:

$$\frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\sin\alpha$$

Rules of Partial Differentiation

The rules of partial differentiation are very similar to those of ordinary differentiation

- **Rule 1 (scaler multiplication rule)**
  If k is a constant then

$$\frac{\partial}{\partial x}(kf) = k\frac{\partial f}{\partial x} \qquad \frac{\partial}{\partial y}(kf) = k\frac{\partial f}{\partial y}$$

- **Rule 2 (sum rule)**
  If $u = f(x,y)$ and $v = g(x,y)$ then

$$\frac{\partial}{\partial x}(u+v) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \qquad \frac{\partial}{\partial y}(u+v) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

- **Rule 3 (product rule)**
  If $u = f(x,y)$ and $v = g(x,y)$ then

$$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} \qquad \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$$

Partial Differentiation

Example:
Given the function $f(x,y) = x^2y^3 + 3y + x$, determine its partial derivative with respect to x and y. Hence determine its directional derivative for a direction at angle $\alpha = \pi/4$ to the x axis.
Solution:
To find the partial derivative of $f(x,y)$ with respect to x, we differentiate $f(x,y)$ and keep y constant

$$\frac{\partial f}{\partial x} = 2xy^3 + 1$$

The partial derivative of $f(x,y)$ with respect to y, by differentiating $f(x,y)$ while keeping x constant

$$\frac{\partial f}{\partial y} = 3x^2y^2 + 3$$

The directional derivative is

$$(2xy^3 + 1)\cos\alpha + (3x^2y^2 + 3)\sin\alpha = \frac{\sqrt{2}}{2}(2xy^3 + 3x^2y^2 + 4)$$

Partial Differentiation

Example:
Given the function $f(x,y) = x^2y^3 + 3y + x$, determine its partial derivative with respect to x and y. Hence determine its directional derivative for a direction at angle $\alpha = \pi/4$ to the x axis.
Solution:
To find the partial derivative of $f(x,y)$ with respect to x, we differentiate $f(x,y)$ and keep y constant

$$\frac{\partial f}{\partial x} = 2xy^3 + 1$$

The partial derivative of $f(x,y)$ with respect to y, by differentiating $f(x,y)$ while keeping x constant

$$\frac{\partial f}{\partial y} = 3x^2y^2 + 3$$

The directional derivative is

$$(2xy^3 + 1)\cos\alpha + (3x^2y^2 + 3)\sin\alpha = \frac{\sqrt{2}}{2}(2xy^3 + 3x^2y^2 + 4)$$

Rules of Partial Differentiation Continued

- **Rule 4 (quotient rule)**
  If $u = f(x,y)$ and $v = g(x,y)$ then

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\left(\frac{\partial u}{\partial x}\right) - u\left(\frac{\partial v}{\partial x}\right)}{v^2} \qquad \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\left(\frac{\partial u}{\partial y}\right) - u\left(\frac{\partial v}{\partial y}\right)}{v^2}$$

- **Rule 5 (simple composite–functions or simple chain rule)**
  If $u = g(f(x,y)) = g(w)$ then

$$\frac{\partial u}{\partial x} = \frac{\mathrm{d}g}{\mathrm{d}w}\frac{\partial w}{\partial x} \qquad \frac{\partial u}{\partial y} = \frac{\mathrm{d}g}{\mathrm{d}w}\frac{\partial w}{\partial y}$$

  we will later extend this concept to where g can not be written just in terms of a single vairable w.

Partial Differentiation

Example: Determine $\partial f/\partial x$ and $\partial f/\partial y$ when $f(x,y)$ is

a) $x^2y^2 + 3xy - x + 2$ b) $\sin(x^2 - 3y)$

Solution:

a) For $f(x,y) = x^2y^2 + 3xy - x + 2$ we have

$$\frac{\partial f}{\partial x} = 2xy^2 + 3y - 1 \qquad \frac{\partial f}{\partial y} = 2x^2y + 3x$$

b) For $f(x,y) = \sin(x^2 - 3y) = \sin w$ with $w = x^2 + y^2$ we have

$$\frac{\partial f}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}w}\frac{\partial w}{\partial x} = \cos(x^2 - 3y)\frac{\partial}{\partial x}(x^2 - 3y) = 2x\cos(x^2 - 3y)$$

$$\frac{\partial f}{\partial y} = -3\cos(x^2 - 3y)$$

Generalising Partial Differentiation

We've already seen partial differentiation applied to functions of two variables.

The concept may be extended to functions of many variables.

For a function $f(x_1, x_2, \cdots x_n)$ of n variables, the partial derivative with respect to x_i is

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \cdots, x_i + \Delta x_i, x_{i+1}, \cdots, x_n) - f(x_1, x_2, \cdots, x_i, x_{i+1}, \cdots, x_n)}{\Delta x_i}$$

Differentiating the function with respect to x_i while keeping all other $n-1$ variables constant.

Generalising Partial Differentiation

Example: Determine $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ when

$$f(x,y,z) = xyz^2 + 3xy - z$$

Solution: We obtain that

$$\frac{\partial f}{\partial x} = yz^2 + 3y$$
$$\frac{\partial f}{\partial y} = xz^2 + 3x$$
$$\frac{\partial f}{\partial x} = 2xyz - 1$$

Generalising the Chain Rule

We've already seen a particular case of the the chain rule in partial differentiation, now lets generalise it.

Let $z = g(v, w)$ and v and w are themselves functions of two independent variables x and y. z is a function of x and y, say $G(x, y)$. We want to differentiate z with respect to x or y

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x} \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial v}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$$

In matrix notation

$$\begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial z}{\partial v} \\ \frac{\partial z}{\partial w} \end{pmatrix}$$

This is the **chain rule** for a function of a two variables.

If $z = g(w)$ and $w = f(x, y)$ this reduces to the simpler case previously presented.

Chain Rule I

Example: Find $\partial T/\partial r$ and $\partial T/\partial \theta$ when

$$T(x,y) = x^2 + 2xy + y^3x^2$$

and $x = r\cos\theta$ and $y = r\sin\theta$
Solution: By the chain rule

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r}$$

In this example

$$\frac{\partial T}{\partial x} = 2x + 2y + 2xy^3 \qquad \frac{\partial T}{\partial y} = 2x + 3x^2y^2$$

and

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

so that

$$\begin{aligned}\frac{\partial T}{\partial r} &= (2x + 2y + 2xy^3)\cos\theta + (2x + 3x^2y^2)\sin\theta \\ &= (2r\cos\theta + 2r\sin\theta + 2r^4\cos\theta\sin^3\theta)\cos\theta \\ &+ (2r\cos\theta + 3r^4\cos^2\theta\sin^2\theta)\sin\theta\end{aligned}$$

Chain Rule II

Similarly

$$
\begin{aligned}
\frac{\partial T}{\partial \theta} &= -(2x + 2y + 2xy^3) r \sin\theta + (2x + 3x^2y^2) r \cos\theta \\
&= -(2r\cos\theta + 2r\sin\theta + 2r^4\cos\theta\sin^3\theta) r\sin\theta \\
&+ (2r\cos\theta + 3r^4\cos^2\theta\sin^2\theta) r\cos\theta
\end{aligned}
$$

Chain Rule I

Example: Find $dR/\,ds$ when

$$R(x,y) = \cosh(x^2 + 3y)$$

and $x(s) = s^2 + 3s$ and $y(s) = \sin s$.
Solution: For this example, x and y are functions of s only so

$$\frac{dR}{ds} = \frac{\partial R}{\partial x}\frac{dx}{ds} + \frac{\partial R}{\partial y}\frac{dy}{ds}$$

which gives

$$\begin{aligned}
\frac{dR}{ds} &= 2x(2s+3)\sinh(x^2+3y) + 3\cos s \sinh(x^2+3y) \\
&= 2(s^2+3s)(2s+3)\sinh((s^2+3s)^2 + 3\sin s) \\
&+ 3\cos s \sinh((s^2+3s)^2 + 3\sin s) \\
&= 2(2s^3+9s^2+9s)\sinh((s^2+3s)^2 + 3\sin s) \\
&+ 3\cos s \sinh((s^2+3s)^2 + 3\sin s)
\end{aligned}$$

Outline

Higher order partial derivatives

We have considered functions like $f(x, y)$ and found its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. If the partial derivatives are also functions of x and y, they can also be differentiated with respect to x and y. We define higher order partial derivatives as follows

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Higher Order Partial Derivatives

If $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are continuous, then it follows that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

If the conditions are not fulfilled these so called **mixed partial derivatives** are not equal.

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$$

Higher Order Partial Derivatives I

Example: For the function

$$f(x,y) = \sin x \cos y + x^3 e^y$$

find all the second order partial derivatives.

Solution: First we find the first order partial derivatives

$$\frac{\partial f}{\partial x} = \cos x \cos y + 3x^2 e^y \qquad \frac{\partial f}{\partial y} = -\sin x \sin y + x^3 e^y$$

Then by differentiating these expressions again we can find the second order derivatives

$$\frac{\partial^2 f}{\partial x^2} = -\sin x \cos y + 6xe^y \qquad \frac{\partial^2 f}{\partial y^2} = -\sin x \cos y + x^3 e^y$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\cos x \sin y + 3x^2 e^y = \frac{\partial^2 f}{\partial y \partial x}$$

In this case, we have that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Higher Order Partial Derivatives I

Example: Find all second partial derivatives of $z = \sin(xy)$.

Solution: First of all the first partial derivatives are found.

$$\frac{\partial^2 z}{\partial x} = y\cos(xy) \qquad \frac{\partial^2 z}{\partial y} = x\cos(xy)$$

Then each of these is differentiated with respect to x:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = -y^2\sin(xy)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -xy\sin(xy) + \cos(xy)$$

Note here the need to use the product rule to differentiate $x\cos(xy)$ with respect to x. Also

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -xy\sin(xy) + \cos(xy) = \frac{\partial^2 z}{\partial x \partial y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = -y^2\sin(xy)$$

Outline

Total differentiation

Let us define: Function $z = f(x, y)$ — function of two variables x and y. Δx is a small change in x, Δy a small change in y and Δz a small change in z. It follows that

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

Rewrite this as the sum of two terms:

1. the first of which shows the change in z due to a change in x
2. the second which shows the change in z due to a change in y

$$\Delta z = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]$$

Total Differentiation

Multiply the first term by $\Delta x/\Delta x = 1$ and the second term by $\Delta y/\Delta y = 1$

$$\Delta z = \frac{[f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)]}{\Delta x}\Delta x + \\ + \frac{[f(x, y+\Delta y) - f(x,y)]}{\Delta y}\Delta y$$

By letting Δx, Δy and $\Delta z \to 0$, we get

$$\mathrm{d}z = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$$

In this expression $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ are called **differentials**.

Total Differentiation

If $z = f(x)$ so that it is a function of one variable, the formula takes the form

$$\mathrm{d}z = \frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x$$

If $w = f(x, y, z)$ is a function of three variables we have

$$\mathrm{d}w = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$$

We can use differentials to calculate errors. If $z = f(x, y)$ and Δx and Δy are errors in x and y, then the error in z is approximately given by

$$\Delta z \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

Total Differentiation I

Example: We want to estimate $\sqrt{(3.01)^2 + (3.97)^2}$

Solution: Let $z = f(x, y) = \sqrt{x^2 + y^2}$.
If we set $x = 3$ and $y = 4$ we can easily compute $z = \sqrt{3^2 + 4^2} = 5$.
Now $\sqrt{(3.01)^2 + (3.97)^2}$ is z when x is increased by $\Delta x = 0.01$ and when y is decreased by 0.03, i.e., $\Delta y = -0.03$

$$\begin{aligned}
\Delta z &\approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \\
&= \frac{1}{2}2x(x^2 + y^2)^{-1/2}\Delta x + \frac{1}{2}2y(x^2 + y^2)^{-1/2}\Delta y \\
&= \frac{x}{\sqrt{x^2 + y^2}}\Delta x + \frac{y}{\sqrt{x^2 + y^2}}\Delta y \\
&= \left(\frac{3}{5} \times 0.01\right) + \frac{4}{5} \times (-0.03) = -0.018
\end{aligned}$$

So $\sqrt{(3.01)^2 + (3.97)^2} \approx 5 + \Delta z = 5 - 0.018 \approx 4.98$.

Total Differentiation I

Example: The height of a cylinder is under measured by 3% and the raduis is over measured by 2% we wish to estimate the percentage error in the volume.

Solution: The volume of a cylinder is given by $V = \pi r^2 h$ as

$$\frac{\partial V}{\partial r} = 2\pi r h \qquad \frac{\partial V}{\partial h} = \pi r^2$$

The error in the volume may be written as

$$\begin{aligned} \Delta V &\approx \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h \\ &= 2\pi r h \Delta r + \pi r^2 \Delta h \end{aligned}$$

As we are interested in the percentage error, we divide this by V

$$\begin{aligned} \Delta V &\approx \frac{2\pi r h}{\pi r^2 h}\Delta r + \frac{\pi r^2}{\pi r^2 h}\Delta h \\ &= \frac{2\Delta r}{r} + \frac{\Delta h}{h} \end{aligned}$$

From the question we know that $\frac{\Delta r}{r} = \frac{2}{100}$ and $\frac{\Delta h}{h} = -\frac{3}{100}$ giving $\frac{\Delta V}{V} = \frac{1}{100}$. This means that the volume is overestimated by 1%.

Total Differentiation I

Example: Find the total differential coefficient of

$$x^2y$$

with respect to x, when x and y are connected by the relation:

$$x^2 + xy + y^2 = 1$$

Solution: Let define $u = x^2y$, then the total differential is

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y$$

Thus the total differential coefficient of u with respect to x is

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x}$$

Total Differentiation II

From the implicit relation $f = x^2 + xy + y^2 = 1$, we first find $f_x = \frac{\partial f}{\partial x} = 2x + y$ and $f_y = \frac{\partial f}{\partial y} = 2y + x$ and see that

$$\frac{f_x}{f_y} = \frac{2x + y}{x + 2y}$$

but, also using the techniques of implicit differentiation from EG189 we have in this case

$$2x + y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x - y}{x + 2y} = -\frac{f_x}{f_y}$$

so that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy + x^2\left[-\frac{2x + y}{x + 2y}\right]$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2xy - \frac{x^2(2x + y)}{x + 2y}.$$

Outline

Integration

In science and engineering: three types of integrals commonly arsie: line integrals, surface integrals and volume integrals.

Line integrals

Are used for computing induced voltage in a coil, mass of wire, work done by a force moving in a vector field...

The basic form of a line integral is

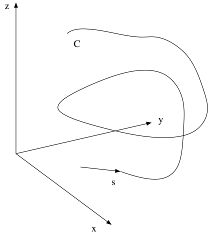
$$\int_C f(x, y, z)\, \mathrm{d}s$$

here s is measured around the contour (possibly a curve or a straight line) C over which the integral is to be performed.

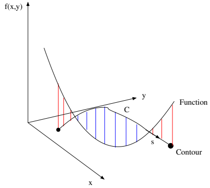
The contour can exist at some arbitrary location in space and rather than integrate than integrate with respect to x as in EG189 in $\int_a^b f(x)\mathrm{d}x$ we need to integrate $f(x, y, z)$ with respect to s along C.

Line Integrals

We don't usually specify limits for line integrals as the starting and end points for the integral are defined by the choice of contour C, which can be arbitrary:



When performing contour integrals, we are working out the area below the function when restricted to the curve C as illustrated below for the case of $f(x, y)$



Line Integrals

Note that in Cartesian coordinates $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$ and in two–dimensions this simplifies to

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1}\mathrm{d}x = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\mathrm{d}y$$

provided s increases as x (resp. y) increases, with the different arrangements offering advantages for lines aligned with the x or y axes repectively.

Example

Compute the line integral of $f(x, y) = x^2 + y$ along the contour defined by the line $y = 2x$ between the points $(-1, -2)$ and $(1, 2)$.

Solution

$$\int_C f(x, y)\mathrm{d}s = \int_C x^2 + y\mathrm{d}s \qquad \mathrm{d}s = \sqrt{(2)^2 + 1}\mathrm{d}x$$

since $\mathrm{d}y/\mathrm{d}x = 2$. Thus

$$\int_{-1}^{1} \left(x^2 + 2x\right) \sqrt{5}\mathrm{d}x = \sqrt{5}\left[\frac{x^3}{3} + x^2\right]_{-1}^{1} = 2\sqrt{5}$$

Working with Parameterisations

However, particular care must be exercised with setting $\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1}\mathrm{d}x$ if s increasing as x is decreasing and similarly with $\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\mathrm{d}y$ with s increasing and y decreasing. in such cases we need the negative root

Example
Find $\int_C \mathrm{d}s$ for the curve $y = x$ between a line starting at $(1,1)$ and finishing at $(0,0)$
Solution
The solution is just the length of the line between the two points i.e. $\sqrt{2}$.

A first approach might be to say $\mathrm{d}y = \mathrm{d}x$ and so $\mathrm{d}s = \sqrt{2}\mathrm{d}x$ and since we go from $(1,1)$ to $(0,0)$ then to say

$$\int_C \mathrm{d}s = \int_1^0 \sqrt{2}\mathrm{d}x = -\sqrt{2}[x]_1^0 = -\sqrt{2}$$

but this has the sign!

The trick is to use the negative root (as s is increasing as x is decreasing) so that $\mathrm{d}s = -\sqrt{2}\mathrm{d}x$

$$\int_0^{\sqrt{2}} \mathrm{d}s = -\int_1^0 \sqrt{2}\mathrm{d}x = \int_0^1 \sqrt{2}\mathrm{d}x = \sqrt{2}[x]_0^1 = \sqrt{2}$$

Working with Parameterisations

However, particular care must be exercised with setting $\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1}\mathrm{d}x$ if s increasing as x is decreasing and similarly with $\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\mathrm{d}y$ with s increasing and y decreasing. in such cases we need the negative root

Example
Find $\int_C \mathrm{d}s$ for the curve $y = x$ between a line starting at $(1,1)$ and finishing at $(0,0)$
Solution
The solution is just the length of the line between the two points i.e. $\sqrt{2}$.

A first approach might be to say $\mathrm{d}y = \mathrm{d}x$ and so $\mathrm{d}s = \sqrt{2}\mathrm{d}x$ and since we go from $(1,1)$ to $(0,0)$ then to say

$$\int_C \mathrm{d}s = \int_1^0 \sqrt{2}\mathrm{d}x = -\sqrt{2}[x]_1^0 = -\sqrt{2}$$

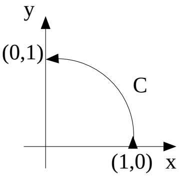
but this has the sign!

The trick is to use the negative root (as s is increasing as x is decreasing) so that $\mathrm{d}s = -\sqrt{2}\mathrm{d}x$

$$\int_0^{\sqrt{2}} \mathrm{d}s = -\int_1^0 \sqrt{2}\mathrm{d}x = \int_0^1 \sqrt{2}\mathrm{d}x = \sqrt{2}[x]_0^1 = \sqrt{2}$$

Line Integrals

Example: Evaluate $\int_C xy \, \mathrm{d}s$ from $(1,0)$ to $(0,1)$ along the curve C that is the portion of $x^2 + y^2 = 1$ in the first quadrant.



Solution:

By implicit differentiation $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. Thus

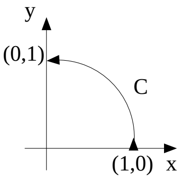
$$\mathrm{d}s = -\sqrt{\frac{x^2}{y^2} + 1}\mathrm{d}x = -\sqrt{\frac{1}{1-x^2}}\mathrm{d}x$$

since s is increasing as x is decreasing. If we integrate wrt s we go from $(1,0)$ to $(0,1)$ thus

$$\int_{(1,0)}^{(0,1)} xy \, \mathrm{d}s = -\int_1^0 x\sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\mathrm{d}x = \int_0^1 x\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

Line Integrals

Example: Evaluate $\int_C xy \, \mathrm{d}s$ from $(1,0)$ to $(0,1)$ along the curve C that is the portion of $x^2 + y^2 = 1$ in the first quadrant.



Solution:

By implicit differentiation $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. Thus

$$\mathrm{d}s = -\sqrt{\frac{x^2}{y^2} + 1}\mathrm{d}x = -\sqrt{\frac{1}{1-x^2}}\mathrm{d}x$$

since s is increasing as x is decreasing. If we integrate wrt s we go from $(1,0)$ to $(0,1)$ thus

$$\int_{(1,0)}^{(0,1)} xy \, \mathrm{d}s = -\int_1^0 x\sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\mathrm{d}x = \int_0^1 x\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

Line Integrals

Example: Evaluate the integral

$$I = \int_C [(x^2 + 2y)\,\mathrm{d}s + (x + y^2)\,\mathrm{d}s]$$

from $(0,1)$ to $(2,3)$ along the curve C defined by $y = x + 1$

Solution:

Since $y = x + 1$ then $\mathrm{d}y = \mathrm{d}x$ and $\mathrm{d}s = \sqrt{2}\mathrm{d}x$ since in this case as s is increasing x is increasing.

$$\begin{aligned} I &= \int_0^2 [(x^2 + 2(x+1)) + (x + (x+1)^2)]\,\sqrt{2}\mathrm{d}x \\ &= \sqrt{2}\int_0^2 (2x^2 + 5x + 3)\,\mathrm{d}x = \sqrt{2}\left[\frac{2}{3}x^3 + \frac{5}{2}x^2 + 3x\right]_0^2 = \frac{64\sqrt{2}}{3} \end{aligned}$$

Line Integrals

Example: Evaluate the integral

$$I = \int_C [(x^2 + 2y)\,\mathrm{d}s + (x + y^2)\,\mathrm{d}s]$$

from $(0,1)$ to $(2,3)$ along the curve C defined by $y = x + 1$

Solution:

Since $y = x + 1$ then $\mathrm{d}y = \mathrm{d}x$ and $\mathrm{d}s = \sqrt{2}\mathrm{d}x$ since in this case as s is increasing x is increasing.

$$\begin{aligned} I &= \int_0^2 [(x^2 + 2(x+1)) + (x + (x+1)^2)]\,\sqrt{2}\mathrm{d}x \\ &= \sqrt{2}\int_0^2 (2x^2 + 5x + 3)\,\mathrm{d}x = \sqrt{2}\left[\frac{2}{3}x^3 + \frac{5}{2}x^2 + 3x\right]_0^2 = \frac{64\sqrt{2}}{3} \end{aligned}$$

Line Integrals

We've already seen line integrals in the form $\int_C f(x,y)\,\mathrm{d}s$ for Cartesian coordinates. Other forms include:

- $\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$ in 2D Cartesian form
- $\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ in 2D parametric form
- $\mathrm{d}s = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$ in 2D polar form (z constant)

If a line integral is such that the integration is performed around a closed (simple) curve, then we denote this type of integral by $\oint_C \mathrm{d}s$ — is evaluated by travelling around C in an anticlockwise direction.

Another form found when dealing with vector fields e.g. $\boldsymbol{F}(x,y,z)$ is the line integral $\int_C \boldsymbol{F}\cdot\mathbf{ds} = \int_C \boldsymbol{F}\cdot\boldsymbol{\tau}\mathrm{d}s$ where $\mathbf{ds} \equiv \boldsymbol{\tau}\mathrm{d}s$ and $\boldsymbol{\tau}$ is a unit tangent to C whose orientation changes with position.

Line Integrals

Example: Evaluate

$$\int_C (z + x^2 - 2y)\mathrm{d}s$$

along the curve C which is specified parametrically as $x = t^2$, $y = t^2$ and $z = 2t^2$ from $(0,0,0)$ to $(1,1,2)$.
Solution: On the curve C, we generalise $\mathrm{d}s$ to

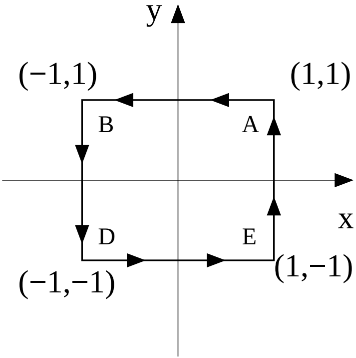
$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \sqrt{4t^2 + 4t^2 + 16t^2}\mathrm{d}t = \sqrt{24}t\mathrm{d}t$$

since $\mathrm{d}x = 2t\mathrm{d}t$, $\mathrm{d}y = 2t\,\mathrm{d}t$ and $\mathrm{d}z = 4t\,\mathrm{d}t$. Also at the point $(0,0,0)$ $t = 0$ and at the point $(1,1,2)$ $t = 1$. In this case as s is increasing as t increasing and so the positive root is correct.

$$\begin{aligned}\int_C (z + x^2 - 2y)\mathrm{d}s &= \sqrt{24}\int_0^1 (2t^2 + t^4 - 2t^2)t\mathrm{d}t \\ &= \sqrt{24}\int_0^1 t^5\,\mathrm{d}t \\ &= \sqrt{24}\left[\frac{t^6}{6}\right]_0^1 = \frac{\sqrt{24}}{6}\end{aligned}$$

Line Integrals I

Example: Evaluate the integral $\oint_C \frac{\mathrm{d}s}{\sqrt{x^2+y^2}}$ where C is the unit square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$.



Line Integrals II

Solution: We can break the integral into four parts

$$\oint_C f(x,y)\mathrm{d}s = \int_{C_{AB}} f(x,y)\mathrm{d}s + \int_{C_{BD}} f(x,y)\mathrm{d}s + \int_{C_{DE}} f(x,y)\mathrm{d}s + \int_{C_{EA}} f(x,y)\mathrm{d}s$$

and use the parameterisation approach:

- Along C_{AB} $y = 1$ and $x(t) = \frac{1}{2}(1-t) - \frac{1}{2}(1+t) = -t$, $-1 \leq t \leq 1$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -1, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 0, \qquad \mathrm{d}s = \sqrt{(-1)^2 + 0^2}\mathrm{d}t = \mathrm{d}t$$

$$\int_{C_{AB}} \frac{\mathrm{d}s}{\sqrt{x^2+1}} = \int_{-1}^{1} \frac{\mathrm{d}t}{\sqrt{t^2+1}}$$

- Along C_{BD} $x = -1$ and $y(t) = -t$, $-1 \leq t \leq 1$

$$\int_{C_{BD}} \frac{\mathrm{d}s}{\sqrt{y^2+1}} = \int_{-1}^{1} \frac{\mathrm{d}t}{\sqrt{t^2+1}}$$

- Along C_{DE} $y = -1$ and $x(t) = t$, $-1 \leq t \leq 1$

$$\int_{C_{DE}} \frac{\mathrm{d}s}{\sqrt{x^2+1}} = \int_{-1}^{1} \frac{\mathrm{d}t}{\sqrt{t^2+1}}$$

Line Integrals III

- Along C_{EA} $x = 1$ and $y(t) = t$, $-1 \le t \le 1$

$$\int_{C_{EA}} \frac{\mathrm{d}s}{\sqrt{y^2+1}} = \int_{-1}^{1} \frac{\mathrm{d}t}{\sqrt{t^2+1}}$$

Note that each part the pameterisation t increases as s increases. Thus the integral becomes

$$\begin{aligned}\oint_C \frac{ds}{\sqrt{x^2+y^2}} &= 4\int_{-1}^{1} \frac{\mathrm{d}t}{\sqrt{1+t^2}} = 4\left[\sinh^{-1} t\right]_{-1}^{1} \\ &= 8\sinh^{-1} 1\end{aligned}$$

Outline

Surface Integrals

Recall $\int_a^b f(x)\mathrm{d}x$ is equal to the area under the curve $f(x)$ between $x = a$ and $x = b$

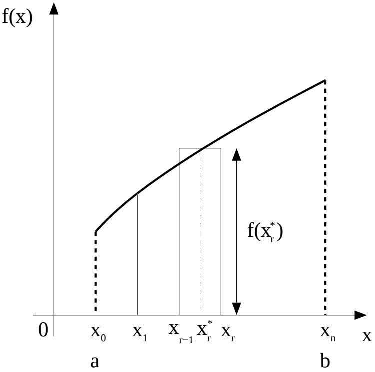


Figure: Integral of a function of a single variable

Surface integrals

We would like have integrals of functions of more than one variable.
Let's consider $z = f(x, y)$ and a region R of the xy plane:

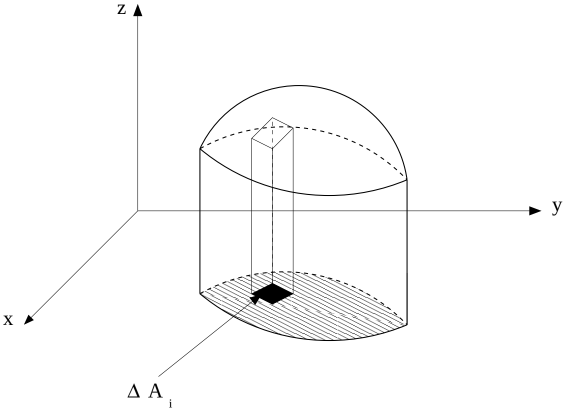


Figure: Integral of a function of two variables.

Surface integrals

We define the integral of $f(x, y)$ over R

$$\int\int_R f(x,y)\,dA = \lim_{\substack{n \to \infty \\ \Delta A_i \to 0}} \sum_{i=1}^{n} f(\bar{x}_i, \bar{y}_i)\Delta A_i$$

where ΔA_i is an elemental area of R and $(\bar{x}_i, \bar{y}_i)$ is a point in ΔA_i.

$f(x, y)$ represents a surface and $f(\bar{x}_i, \bar{y}_i)\Delta A_i = \bar{z}_i \Delta A_i$ is the volume between the $z = 0$ and $z = \bar{z}_i$ whose base cross-section is ΔA_i.

The integral is the limit of the sum of all such volumes and so it is the volume under the surface of $z = f(x, y)$ above R.

Surface Integrals

If we introduce a series of lines which are parallel to the x and y axis:

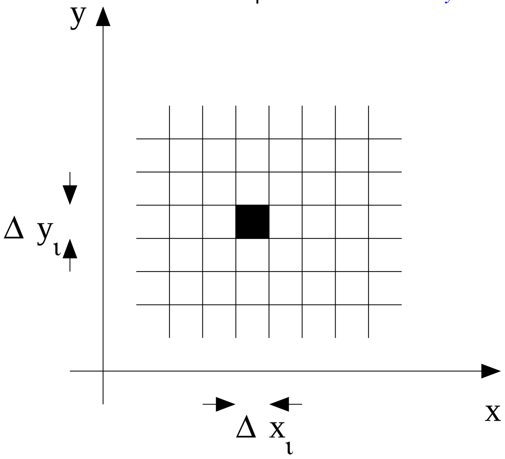


Figure: Lines introduced for double integrals.

Surface Integrals

We can write $\Delta A_i = \Delta x_i \Delta y_i$, giving

$$\int\int_R f(x,y)\,dA = \int\int_R f(x,y)\,dx\,dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(\bar{x}_i,\bar{y}_i)\Delta x_i \Delta y_i$$

Integrals of the type $\int\int_R f(x,y)\,dx\,dy$ can be evaluated as repeated single integrals in x and y. They are usually called **double integrals**.

For the particular case of the integral $\int\int_R dA$ — this equals to the area of region R, ie integration can be used to find out the area of any shape!

Surface Integrals

Two alternatives to evaluating double integrals.
If data is given as $y = g(x)$, i.e., y is some function of x, then we work out the integral by first performing the integration with respect to y and then with respect to x, i.e.

$$\int\int_R f(x,y)\,dA = \int_a^b \left[\int_{y=g_1(x)}^{x=g_2(x)} f(x,y)\,dy\right]\,dx$$

Alternatively, if we have that x is expressed as some function of y, e.g.,$x = h(y)$, then we first perform the integration with respect to x and then integrate with respect to y

$$\int\int_R f(x,y)\,dA = \int_c^d \left[\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\right]\,dy$$

Surface Integrals

In the particular case where the region R is a rectangle, then the limits of the integration are constant and so it does not matter whether integrate x or y first.

$$\int\int_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Other elemental surface differentials include

- $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ Cartesian coordinates, z constant.
- $\mathrm{d}A = \mathrm{d}\rho\mathrm{d}z$ Cylinderical coordinates, ϕ constant
- $\mathrm{d}A = \rho\mathrm{d}\rho\mathrm{d}\phi$ Cylinderical coordinates, z constant
- $\mathrm{d}A = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$ Spherical coordinates, r constant
- $\mathrm{d}A = r\sin\theta\mathrm{d}r\mathrm{d}\phi$ Spherical coordinates, θ constant
- $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$ Spherical coordinates, ϕ constant

and apart from simple rectangular/cylindrical/spherical geometries we need to take care with the order of integration.

Surface Integrals I

Example: Evaluate the integral

$$\int_0^1 \int_1^3 (x^2 + y^2)\, dx\, dy$$

Solution: If we integrate with respect to x first, then we obtain

$$\int_0^1 \int_1^3 (x^2 + y^2)\, dx\, dy = \int_0^1 \left[\frac{1}{3}x^3 + y^2 x\right]_{x=1}^{x=3} dy$$

$$= \int_0^1 \left(\frac{26}{3} + 2y^2\right) dy = \left[\frac{26}{3}y + \frac{2}{3}y^3\right]_0^1 = \frac{28}{3}$$

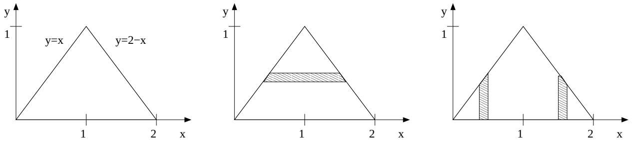
Alternatively with respect to y first

$$\int_0^1 \int_1^3 (x^2 + y^2)\, dx\, dy = \int_1^3 \left[x^2 y + \frac{1}{3}y^3\right]_{y=0}^{y=1} dx$$

$$= \int_1^3 \left(x^2 + \frac{1}{3}\right) dx = \frac{28}{3}$$

Surface Integrals I

Example: Evaluate $\int\int_R (x^2+y^2)\,dA$ over a triangle with vertices $(0,0)$, $(2,0)$ and $(1,1)$.
Solution:



First, integrating with respect to x first gives

$$\int\int_R (x^2+y^2)\,dA = \int_0^1 \int_{x=y}^{x=2-y} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{1}{3}x^3 + y^2 x\right]_{x=y}^{x=2-y} dy = \int_0^1 \left(\frac{8}{3} - 4y + 4y^2 - \frac{8}{3}y^3\right) dy = \frac{4}{3}$$

Surface Integrals II

Next integrating with respect to y first

$$\int\int_R (x^2+y^2)\,dA = \int_0^1 \int_{y=0}^{y=x} (x^2+y^2)\,dy\,dx +$$

$$+\int_1^2 \int_{y=0}^{y=2-x} (x^2+y^2)\,dy\,dx$$

Here the integrals are

$$\int_0^1 \int_{y=0}^{y=x} (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2y+\frac{1}{3}y^3\right]_{y=0}^{y=x} dx =$$

$$= \int_0^1 \frac{4}{3}x^3\,dx = \frac{1}{3}$$

$$\int_1^2 \int_{y=0}^{y=2-x} (x^2+y^2)\,dy\,dx = \int_1^2 \left[x^2y+\frac{1}{3}y^3\right]_{y=0}^{y=2-x} dx =$$

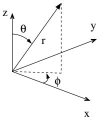
$$= \int_1^2 \left(\frac{8}{3}-4x+4x^2-\frac{4}{3}x^3\right) dx = 1$$

So $\int\int_R (x^2+y^2)\,dA = 1+\frac{1}{3} = \frac{4}{3}$.

Surface Integrals I

Example:
By using a surface integral, check that that surface area of a sphere of radius $r = a$ is $4\pi a^2$

Solution:
In this case we want to find $\int \mathrm{d}A$ and is best to use spherical coordinates:



On A $r = a$ is fixed and so $\mathrm{d}A = r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi = a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ so

$$\text{area} = \int \mathrm{d}A = \int_0^{2\pi} \int_0^{\pi} a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi = a^2 \int_0^{2\pi} \left(\int_0^{\pi} \sin\theta \mathrm{d}\theta \right) \mathrm{d}\phi$$

$$= a^2 \int_0^{2\pi} \left[-\cos\theta \right]_0^{\pi} \mathrm{d}\phi = a^2 \int_0^{2\pi} 2 \mathrm{d}\phi = 4\pi a^2$$

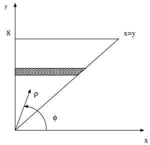
since for a complete sphere $0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$.

Surface Integrals

Example:

Change the integral $\int_0^8 \int_0^y x\mathrm{d}x\mathrm{d}y$ in to an equivalent polar integral and evaluate

Solution:



The region of integration is a triangular region. Recall the transformations

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad \mathrm{d}A = \rho\mathrm{d}\rho\mathrm{d}\phi$$

For this region we need to integrate from $\phi = \pi/4$ to $\phi = \pi/2$, but the radial integration depends on the angle ϕ, for each ϕ we need to integrate from $\rho = 0$ to $8/\sin\phi$.

$$\int_0^8 \int_0^y x\mathrm{d}x\mathrm{d}y = \int_{\pi/4}^{\pi/2} \int_0^{8/\sin\phi} \rho\cos\phi\rho\mathrm{d}\rho\mathrm{d}\phi = \int_{\pi/4}^{\pi/2} \cos\phi \left[\frac{\rho^3}{3}\right]_0^{8/\sin\phi} \mathrm{d}\phi$$

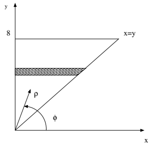
$$= \int_{\pi/4}^{\pi/2} \frac{512\cos\phi}{3\sin^3\phi}\mathrm{d}\phi = -\frac{512}{6}\left[\sin^{-2}\phi\right]_{\pi/4}^{\pi/2} = \frac{256}{3}$$

Surface Integrals

Example:

Change the integral $\int_0^8 \int_0^y x\mathrm{d}x\mathrm{d}y$ in to an equivalent polar integral and evaluate

Solution:



The region of integration is a triangular region. Recall the transformations

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad \mathrm{d}A = \rho\mathrm{d}\rho\mathrm{d}\phi$$

For this region we need to integrate from $\phi = \pi/4$ to $\phi = \pi/2$, but the radial integration depends on the angle ϕ, for each ϕ we need to integrate from $\rho = 0$ to $8/\sin\phi$.

$$\int_0^8 \int_0^y x\mathrm{d}x\mathrm{d}y = \int_{\pi/4}^{\pi/2} \int_0^{8/\sin\phi} \rho\cos\phi\rho\mathrm{d}\rho\mathrm{d}\phi = \int_{\pi/4}^{\pi/2} \cos\phi\left[\frac{\rho^3}{3}\right]_0^{8/\sin\phi} \mathrm{d}\phi$$

$$= \int_{\pi/4}^{\pi/2} \frac{512\cos\phi}{3\sin^3\phi}\mathrm{d}\phi = -\frac{512}{6}\left[\sin^{-2}\phi\right]_{\pi/4}^{\pi/2} = \frac{256}{3}$$

Volume Integrals

Volume integrals — three successive integrals.
Volume integrals are of the form:

$$\int\int\int_V dV$$

and are called **triple integrals**.

Evaluated in the same way as double integrals — we start by evaluating the inner integral and work outwards.

The main difficulty — to determine the correct limits for the integration — make a sketch of the region.

The differential $\mathrm{d}V$ is

- $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ for Cartesian coordinates
- $\mathrm{d}V = \rho\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}z$ for Cylindrical coordinates
- $\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta$ for Spherical coordinates

In the Cartesian case, if the integrals are evaluated in the order x, y, z then the limits on the y integral may depend on z but not on x with similar restrictions for cylindrical and spherical coordinates.

Volume Integrals I

Example: A cube $0 \leq x, y, z \leq 1$m has a variable density given by $\rho = 1 + x + y + z$ kg/m^3, what is the total mass of the cube
Solution: The total mass is given by

$$M = \int\int\int_V \rho \, dV =$$

$$= \int_0^1 \int_0^1 \int_0^1 (1 + x + y + z) \, dx \, dy \, dz =$$

$$= \int_0^1 \int_0^1 \left[x + \frac{x^2}{2} + xy + xz \right]_0^1 dy \, dz$$

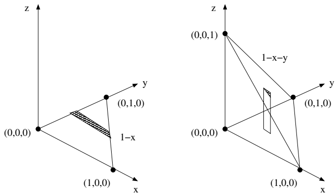
$$= \int_0^1 \int_0^1 \left(\frac{3}{2} + y + z \right) dy \, dz =$$

$$= \int_0^1 \left[\frac{3y}{2} + \frac{y^2}{2} + yz \right]_0^1 dz$$

$$= \int_0^1 (2 + z) \, dz = \left[2z + \frac{z^2}{2} \right]_0^1 = \frac{5}{2}\text{kg}$$

Volume Integrals

Example: A tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ (m) in Cartesian coordinates has density $\rho = 10x$ kg/m^3, what is its total mass?
Solution: Consider the sketches



For the triangular base, we integrate x from 0 to $1-y$ and y from 0 to 1. For z we integrate from 0 to $1-x-y$.

This is since the plane containing the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ is

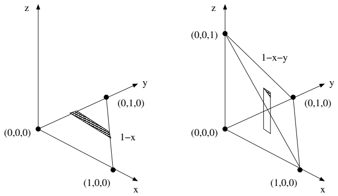
$$\boldsymbol{n}\cdot\boldsymbol{w} = \boldsymbol{n}\cdot\boldsymbol{c} \qquad \boldsymbol{n} = \boldsymbol{a}\times\boldsymbol{b}$$

and $\boldsymbol{a} = -1\boldsymbol{e}_1 + 1\boldsymbol{e}_3$ and $\boldsymbol{b} = -1\boldsymbol{e}_2 + 1\boldsymbol{e}_3$ are two vectors in the plane and $\boldsymbol{c} = \boldsymbol{e}_3$ a vector pointing to a point in the plane. Thus $\boldsymbol{n} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ and

$$\boldsymbol{n}\cdot\boldsymbol{w} = x + y + z = 1 \qquad z = 1 - x - y$$

Volume Integrals

Example: A tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ (m) in Cartesian coordinates has density $\rho = 10x$ kg/m^3, what is its total mass?
Solution: Consider the sketches



For the triangular base, we integrate x from 0 to $1-y$ and y from 0 to 1. For z we integrate from 0 to $1-x-y$.

This is since the plane containing the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ is

$$\boldsymbol{n}\cdot\boldsymbol{w} = \boldsymbol{n}\cdot\boldsymbol{c} \qquad \boldsymbol{n} = \boldsymbol{a}\times\boldsymbol{b}$$

and $\boldsymbol{a} = -1\boldsymbol{e}_1 + 1\boldsymbol{e}_3$ and $\boldsymbol{b} = -1\boldsymbol{e}_2 + 1\boldsymbol{e}_3$ are two vectors in the plane and $\boldsymbol{c} = \boldsymbol{e}_3$ a vector pointing to a point in the plane. Thus $\boldsymbol{n} = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$ and

$$\boldsymbol{n}\cdot\boldsymbol{w} = x + y + z = 1 \qquad z = 1 - x - y$$

Volume Integrals

Thus

$$\text{mass} = \int_0^1 \int_0^{1-y} \int_0^{1-x-y} \rho \, dz dx dy = 10 \int_0^1 \left(\int_0^{1-y} x \left(\int_0^{1-x-y} dz \right) dx \right) dy$$

$$= 10 \int_0^1 \left(\int_0^{1-y} x \left[z \right]_0^{1-x-y} dx \right) dy = 10 \int_0^1 \left(\left[x - \frac{x^2}{2} - yx \right]_0^{1-y} \right) dy$$

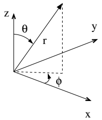
$$= 10 \int_0^1 \frac{(1-y)^2}{2} dy = 10 \left[-\frac{(1-y)^3}{6} \right]_0^1 = \frac{10}{6} \text{ kg}$$

Volume Integrals

Example: Evaluate the integral $\int_H z \mathrm{d}V$ where H is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

This is best done in spherical coordinates.



The transformations from spherical to Cartesian are

$$x = r \sin\theta \cos\phi \qquad y = r \sin\theta \sin\phi \qquad z = r \cos\theta$$

and $\mathrm{d}V = r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi$. For the upper half of the sphere $x^2 + y^2 + z^2 = 1$

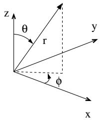
$$0 \leq r \leq 1 \qquad 0 \leq \phi \leq 2\pi \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

Volume Integrals

Example: Evaluate the integral $\int_H z\mathrm{d}V$ where H is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

This is best done in spherical coordinates.



The transformations from spherical to Cartesian are

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

and $\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$. For the upper half of the sphere $x^2 + y^2 + z^2 = 1$

$$0 \leq r \leq 1 \qquad 0 \leq \phi \leq 2\pi \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

Volume Integrals

Thus

$$\int_H z\mathrm{d}V = \int_0^1 \int_0^{2\pi} \int_0^{\pi/2} (4r\cos\theta)(r^2\sin\theta)\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}r = 4\int_0^1 r^3 \int_0^{2\pi} \int_0^{\pi/2} \cos\theta\sin\theta\mathrm{d}\theta\mathrm{d}\phi\mathrm{d}r$$

$$=4\int_0^1 r^3 \int_0^{2\pi} \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \mathrm{d}\phi\mathrm{d}r = 2\int_0^1 r^3\,[\phi]_0^{2\pi} = 4\pi\left[\frac{r^4}{4}\right]_0^1 = \pi$$