Swansea University
Prifysgol Abertawe

Engineering Analysis 1 : Linear Algebra

Dr. Paul D. Ledger, Dr Rob Daniels, Dr Igor Sazonov

engmaths@swansea.ac.uk

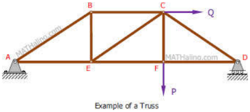
College of Engineering, Swansea University, UK

Outline

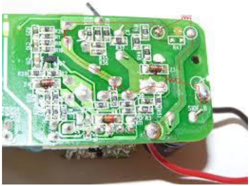
Applications

Being able to work with and solve systems of linear equations is a key skill required for engineers and appears in all areas of engineering and beyond, e.g.

Traffic Modeling

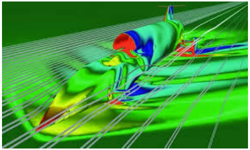


Truss Analysis



Electrical Circuits

Pipe Networks



Flow Simulation

Google

Internet search engines

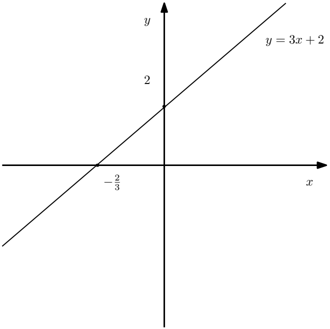
Outline

What is a linear equation?

The simplest **linear equation** is that between two variables that gives a straight line when plotted on a graph

$$y = kx + c$$

Here $k = 3$ is the gradient and the intercept on the y-axis is $c = 2$.



If $k \neq 0$ then $x = (y - c)/k$ is the "solution" to $y = kx + c$ given y.

Simultaneous Equations

Given the equations of two lines with gradients k and ℓ and intercepts c and d:

$$\begin{aligned} y &= kx + c \\ y &= \ell x + d \end{aligned} \qquad \text{or} \qquad \begin{aligned} y - kx &= c \\ y - \ell x &= d \end{aligned}$$

one might to find the point (x, y) (if it exists) where the lines intersect.

This problem is an example of solving a set of 2 **linear** (**simultaneous**) **equations** and 2 unknowns.

Importantly, our unknowns need not be x and y, but can be other letters, symbols or with indices appropriate to the practical problem at hand.

The number of equations and unknowns in a set of simultaneous equations can be more than 2 and need not be equal in general!

Examples

Example

Solve

$$
\begin{aligned}
x + 2y &= 5 \\
2x + 3y &= 8
\end{aligned}
$$

Solution

Two equations and two unknowns. Solution is $x = 1$ and $y = 2$.

Note that in general a linear equation system may have m equations and n unknowns.

Example

Solve

$$
\begin{aligned}
x + y &= 4 \\
2x + 2y &= 5
\end{aligned}
$$

Solution

Here $m = n = 2$ No solution since $2(x + y) = 8$ and $2x + 2y = 5$

Examples

Example
Solve

$$\begin{aligned} x + 2y &= 5 \\ 2x + 3y &= 8 \end{aligned}$$

Solution
Two equations and two unknowns. Solution is $x = 1$ and $y = 2$.
Note that in general a linear equation system may have m equations and n unknowns.

Example
Solve

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 5 \end{aligned}$$

Solution
Here $m = n = 2$ No solution since $2(x + y) = 8$ and $2x + 2y = 5$

Examples

Example
Solve

$$\begin{aligned} x + 2y &= 5 \\ 2x + 3y &= 8 \end{aligned}$$

Solution
Two equations and two unknowns. Solution is $x = 1$ and $y = 2$.
Note that in general a linear equation system may have m equations and n unknowns.

Example
Solve

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 5 \end{aligned}$$

Solution
Here $m = n = 2$ No solution since $2(x + y) = 8$ and $2x + 2y = 5$

Examples

Example
Solve

$$\begin{aligned} x + 2y &= 5 \\ 2x + 3y &= 8 \end{aligned}$$

Solution
Two equations and two unknowns. Solution is $x = 1$ and $y = 2$.
Note that in general a linear equation system may have m equations and n unknowns.
Example
Solve

$$\begin{aligned} x + y &= 4 \\ 2x + 2y &= 5 \end{aligned}$$

Solution
Here $m = n = 2$ No solution since $2(x + y) = 8$ and $2x + 2y = 5$

Examples

Example
Solve

$$\begin{aligned} u - v + w &= 2 \\ 2u + v - w &= 4 \end{aligned}$$

Solution
A possible solution is $u = 2$, $v = 0$ and $w = 0$ another is $u = 2$, $v = 1$ and $w = 1$. In general there are infinitely many solutions, namely $u = 2$, $v = \alpha$ and $w = \alpha$ where α is any real number.

Example
Solve

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 &= 4 \end{aligned}$$

Solution
Here $m = 3$ and $n = 2$. No solution since $x_1 = 3/2$ and $x_1 = 4$.

Examples

Example
Solve

$$\begin{aligned} u - v + w &= 2 \\ 2u + v - w &= 4 \end{aligned}$$

Solution
A possible solution is $u = 2$, $v = 0$ and $w = 0$ another is $u = 2$, $v = 1$ and $w = 1$. In general there are infinitely many solutions, namely $u = 2$, $v = \alpha$ and $w = \alpha$ where α is any real number.

Example
Solve

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 &= 4 \end{aligned}$$

Solution
Here $m = 3$ and $n = 2$. No solution since $x_1 = 3/2$ and $x_1 = 4$.

Examples

Example
Solve

$$\begin{aligned} u - v + w &= 2 \\ 2u + v - w &= 4 \end{aligned}$$

Solution
A possible solution is $u = 2$, $v = 0$ and $w = 0$ another is $u = 2$, $v = 1$ and $w = 1$. In general there are infinitely many solutions, namely $u = 2$, $v = \alpha$ and $w = \alpha$ where α is any real number.

Example
Solve

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 &= 4 \end{aligned}$$

Solution
Here $m = 3$ and $n = 2$. No solution since $x_1 = 3/2$ and $x_1 = 4$.

Examples

Example
Solve

$$\begin{aligned} u - v + w &= 2 \\ 2u + v - w &= 4 \end{aligned}$$

Solution
A possible solution is $u = 2$, $v = 0$ and $w = 0$ another is $u = 2$, $v = 1$ and $w = 1$. In general there are infinitely many solutions, namely $u = 2$, $v = \alpha$ and $w = \alpha$ where α is any real number.

Example
Solve

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 &= 4 \end{aligned}$$

Solution
Here $m = 3$ and $n = 2$. No solution since $x_1 = 3/2$ and $x_1 = 4$.

Outline

Equivalent linear systems

From now one we will label the unknowns as $x_1, x_2, \cdots x_n$ for reasons that will shortly become clear. Note the following operations:

Exchanging equations

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned} \quad \text{is equivalent to} \quad \begin{aligned} 2x_1 + 3x_2 &= 8 \\ x_1 + 2x_2 &= 5 \end{aligned}$$

Both equation systems posses the same solution set.

Addition of factored equation to another equation

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned} \quad \text{is equivalent to} \quad \begin{aligned} \boxed{1}x_1 + 2x_2 &= 5 \\ -x_2 &= -2 \end{aligned}$$

To obtain the revised second equation, we must multiplied the first equation by 2 and then subtracted it from the second equation in the original system. We call the right **upper triangular** form, which makes it easier to solve. Both equation systems posses the same solution set.

cyan indicates a row swap, blue indicates a pivot row with pivot highlighted, red indicates a row after elimination.

Upper triangular form and back substitution

Example

Solve

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

Here $m = n = 3$.

Solution

From the third equation we have $x_3 = 2$. Substituting this in to the second equation gives $x_2 = 4$ and finally using both values in the first equation gives $x_1 = -3$.

Example

$$\begin{aligned} 2x_2 + 2x_3 &= 1 \\ 2x_1 + 4x_2 + 5x_3 &= 9 \\ x_1 - x_2 + 2x_3 &= 3 \end{aligned}$$

Here $m = n = 3$.

Upper triangular form and back substitution

Example
Solve

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

Here $m = n = 3$.

Solution
From the third equation we have $x_3 = 2$. Substituting this in to the second equation gives $x_2 = 4$ and finally using both values in the first equation gives $x_1 = -3$.

Example

$$\begin{aligned} 2x_2 + 2x_3 &= 1 \\ 2x_1 + 4x_2 + 5x_3 &= 9 \\ x_1 - x_2 + 2x_3 &= 3 \end{aligned}$$

Here $m = n = 3$.

Upper triangular form and back substitution

Example
Solve

$$
\begin{aligned}
3x_1 + 2x_2 + x_3 &= 1 \\
x_2 - x_3 &= 2 \\
2x_3 &= 4
\end{aligned}
$$

Here $m = n = 3$.
Solution
From the third equation we have $x_3 = 2$. Substituting this in to the second equation gives $x_2 = 4$ and finally using both values in the first equation gives $x_1 = -3$.
Example

$$
\begin{aligned}
2x_2 + 2x_3 &= 1 \\
2x_1 + 4x_2 + 5x_3 &= 9 \\
x_1 - x_2 + 2x_3 &= 3
\end{aligned}
$$

Here $m = n = 3$.

Upper triangular form and back substitution

Solution
As the coefficient of x_1 is zero in the first equations (it is not a valid **pivot**) so we exchange the first and second equations

$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 &= 9 \\ 2x_2 + 2x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 3 \end{aligned}$$

Next we multiply the first equation by $\frac{1}{2}$ and subtract it from the third equation

$$\begin{aligned} \boxed{2}x_1 + 4x_2 + 5x_3 &= 9 \\ 2x_2 + 2x_3 &= 1 \\ -3x_2 - \frac{1}{2}x_3 &= -\frac{3}{2} \end{aligned}$$

Multiply the second equation by $\frac{3}{2}$ and add it to the third equation, giving

$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 &= 9 \\ \boxed{2}x_2 + 2x_3 &= 1 \\ \frac{5}{2}x_3 &= 0 \end{aligned}$$

By back substitution we find the solution $x_1 = \frac{7}{2}$, $x_2 = \frac{1}{2}$ and $x_3 = 0$. It follows that this is the only solution to the linear system.

Gauss elimination

Consider a system where $m = n = 3$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{13}x_3 &= b_3 \end{aligned}$$

The values a_{ij} are the real **coefficients** of the unknown x_j in the ith equation of the system. The number b_i is the right hand side of the ith equation.

we express the system in the form

x_1	x_2	x_3	1
a_{11}	a_{12}	a_{13}	b_1
a_{21}	a_{22}	a_{23}	b_2
a_{31}	a_{32}	a_{33}	b_3

The Gauss elimination procedure comprises of two stages: elimination and back substitution.

Example

Example

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 + x_4 &= 5 \\
2x_1 + x_2 + x_3 + x_4 &= 3 \\
x_1 + 2x_2 + x_3 &= 4 \\
x_2 + x_3 + 2x_4 &= 0
\end{aligned}
$$

Solution

Write the equation in schematic representation and apply Gauss algorithm

x_1	x_2	x_3	x_4	1
1	2	3	1	5
2	1	1	1	3
1	2	1	0	4
0	1	1	2	0

x_1	x_2	x_3	x_4	1
1	2	3	1	5
0	-3	-5	-1	-7
0	0	-2	-1	-1
0	1	1	2	0

x_1	x_2	x_3	x_4	1
1	2	3	1	5
0	-3	-5	-1	-7
0	0	-2	-1	-1
0	0	$-\frac{2}{3}$	$\frac{5}{3}$	$-\frac{7}{3}$

x_1	x_2	x_3	x_4	1
1	2	3	1	5
0	-3	-5	-1	-7
0	0	-2	-1	-1
0	0	0	$\frac{6}{3}$	$-\frac{6}{3}$

We then determine the solution through back substitution giving $x_4 = -1$, $x_3 = 1$, $x_2 = 1$ and $x_1 = 1$

Example

Example

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &= 5 \\ 2x_1 + x_2 + x_3 + x_4 &= 3 \\ x_1 + 2x_2 + x_3 &= 4 \\ x_2 + x_3 + 2x_4 &= 0 \end{aligned}$$

Solution

Write the equation in schematic representation and apply Gauss algorithm

x_1	x_2	x_3	x_4	1
1	2	3	1	5
2	1	1	1	3
1	2	1	0	4
0	1	1	2	0

x_1	x_2	x_3	x_4	1
1	2	3	1	5
0	-3	-5	-1	-7
0	0	-2	-1	-1
0	1	1	2	0

x_1	x_2	x_3	x_4	1
1	2	3	1	5
0	-3	-5	-1	-7
0	0	-2	-1	-1
0	0	$-\frac{2}{3}$	$\frac{5}{3}$	$-\frac{7}{3}$

x_1	x_2	x_3	x_4	1
1	2	3	1	5
0	-3	-5	-1	-7
0	0	-2	-1	-1
0	0	0	$\frac{6}{3}$	$-\frac{6}{3}$

We then determine the solution through back substitution giving $x_4 = -1$, $x_3 = 1$, $x_2 = 1$ and $x_1 = 1$

Example

Example

$$\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 + 2x_2 &= 5 \end{aligned}$$

Solution

First we write the equation in schematic representation

x_1	x_2	1
1	1	4
2	2	5

Then we proceed with the Gauss elimination algorithm, giving

x_1	x_2	1
1	1	4
0	0	-3

Clearly $0x_1 + 0x_2 \neq -3$ so the linear system has no solution.

Example

Example

$$\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 + 2x_2 &= 5 \end{aligned}$$

Solution

First we write the equation in schematic representation

x_1	x_2	1
1	1	4
2	2	5

Then we proceed with the Gauss elimination algorithm, giving

x_1	x_2	1
1	1	4
0	0	-3

Clearly $0x_1 + 0x_2 \neq -3$ so the linear system has no solution.

Example

Example

Deduce the solution set(s) and how they depend on s for

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - x_4 + x_5 &= -2 \\ 2x_1 - x_2 + 3x_3 - x_5 &= -3 \\ -4x_1 + 2x_2 - 4x_3 + 5x_4 - 5x_5 &= 3 \\ -2x_3 + 2x_4 - 7x_5 &= -5 + s \\ -2x_1 + x_2 - x_3 + 4x_5 &= 5 \end{aligned}$$

Solution

First we write the equation in schematic representation

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
2	-1	3	0	-1	-3
-4	2	-4	5	-5	3
0	0	-2	2	-7	-5+s
-2	1	-1	0	4	5

then we proceed with the Gauss elimination algorithm:

Example

Example

Deduce the solution set(s) and how they depend on s for

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - x_4 + x_5 &= -2 \\ 2x_1 - x_2 + 3x_3 - x_5 &= -3 \\ -4x_1 + 2x_2 - 4x_3 + 5x_4 - 5x_5 &= 3 \\ -2x_3 + 2x_4 - 7x_5 &= -5 + s \\ -2x_1 + x_2 - x_3 + 4x_5 &= 5 \end{aligned}$$

Solution

First we write the equation in schematic representation

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
2	-1	3	0	-1	-3
-4	2	-4	5	-5	3
0	0	-2	2	-7	-5+s
-2	1	-1	0	4	5

then we proceed with the Gauss elimination algorithm:

Example continued

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
0	0	0	1	-2	-1
0	0	2	3	-3	-1
0	0	-2	2	-7	-5+s
0	0	2	-1	5	3

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
0	0	2	3	-3	-1
0	0	0	1	-2	-1
0	0	-2	2	-7	-5+s
0	0	2	-1	5	3

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
0	0	2	3	-3	-1
0	0	0	1	-2	-1
0	0	0	5	-10	-6+s
0	0	0	-4	8	4

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
0	0	2	3	-3	-1
0	0	0	1	-2	-1
0	0	0	0	0	-1+s
0	0	0	0	0	0

For $s \neq 1$ we have no solution. For $s = 1$ we find

$$\begin{aligned} x_4 &= -1 + 2x_5 \\ x_3 &= \frac{1}{2}(-1 + 3x_5 - 3x_4) = 1 - \frac{3}{2}x_5 \\ x_1 &= \frac{1}{2}(-2 - x_5 + x_4 - 3x_3 + x_2) = -3 + \frac{11}{4}x_5 + \frac{1}{2}x_2 \end{aligned}$$

The solution set for $s = 1$ consists of two free parameters β and α and has infinitely many solutions $x_1 = -3 + \frac{11}{4}\beta + \frac{1}{2}\alpha$, $x_2 = \alpha$, $x_3 = 1 - \frac{3}{2}\beta$, $x_4 = -1 + 2\beta$, $x_5 = \beta$.

Linear systems of equations with multiple right hand sides

Example

$$\begin{aligned} 2x_2 + 2x_3 &= b_1 \\ 2x_1 + 4x_2 + 5x_3 &= b_2 \\ x_1 - x_2 + 2x_3 &= b_3 \end{aligned} \qquad a) \begin{aligned} b_1 &= 1 \\ b_2 &= 9 \\ b_3 &= 3 \end{aligned} \qquad b) \begin{aligned} b_1 &= 2 \\ b_2 &= 13 \\ b_3 &= 1 \end{aligned} \qquad c) \begin{aligned} b_1 &= 5 \\ b_2 &= -4 \\ b_3 &= 2 \end{aligned}$$

Solution

x_1	x_2	x_3	1_a	1_b	1_c
0	2	2	1	2	5
2	4	5	9	13	-4
1	-1	2	3	1	2

x_1	x_2	x_3	1_a	1_b	1_c
1	-1	2	3	1	2
2	4	5	9	13	-4
0	2	2	1	2	5

x_1	x_2	x_3	1_a	1_b	1_c
1	-1	2	3	1	2
0	6	1	3	11	-8
0	2	2	1	2	5

x_1	x_2	x_3	1_a	1_b	1_c
1	-1	2	3	1	2
0	6	1	3	11	-8
0	0	$\frac{5}{3}$	0	$-\frac{5}{3}$	$\frac{23}{3}$

By back substitution we obtain the solutions a) $x_1 = \frac{7}{2}$, $x_2 = \frac{1}{2}$, $x_3 = 0$, b) $x_1 = 5$, $x_2 = 2$, $x_3 = -1$ and c) $x_1 = -\frac{279}{30}$, $x_2 = -\frac{63}{30}$, $x_3 = \frac{23}{5}$.

Linear systems of equations with multiple right hand sides

Example

$$\begin{aligned} 2x_2 + 2x_3 &= b_1 \\ 2x_1 + 4x_2 + 5x_3 &= b_2 \\ x_1 - x_2 + 2x_3 &= b_3 \end{aligned} \qquad a) \begin{aligned} b_1 &= 1 \\ b_2 &= 9 \\ b_3 &= 3 \end{aligned} \qquad b) \begin{aligned} b_1 &= 2 \\ b_2 &= 13 \\ b_3 &= 1 \end{aligned} \qquad c) \begin{aligned} b_1 &= 5 \\ b_2 &= -4 \\ b_3 &= 2 \end{aligned}$$

Solution

x_1	x_2	x_3	1_a	1_b	1_c
0	2	2	1	2	5
2	4	5	9	13	-4
1	-1	2	3	1	2

x_1	x_2	x_3	1_a	1_b	1_c
1	-1	2	3	1	2
2	4	5	9	13	-4
0	2	2	1	2	5

x_1	x_2	x_3	1_a	1_b	1_c
1	-1	2	3	1	2
0	6	1	3	11	-8
0	2	2	1	2	5

x_1	x_2	x_3	1_a	1_b	1_c
1	-1	2	3	1	2
0	6	1	3	11	-8
0	0	$\frac{5}{3}$	0	$-\frac{5}{3}$	$\frac{23}{3}$

By back substitution we obtain the solutions a) $x_1 = \frac{7}{2}$, $x_2 = \frac{1}{2}$, $x_3 = 0$, b) $x_1 = 5$, $x_2 = 2$, $x_3 = -1$ and c) $x_1 = -\frac{279}{30}$, $x_2 = -\frac{63}{30}$, $x_3 = \frac{23}{5}$.

Rank of a linear system

The **rank** of a linear equation system is defined as the number, r, of non–zero rows after performing the Gauss elimination algorithm.

Using the rank, the solution set of the linear system of equations can be described: A linear equation system with m equations and n unknowns has at least one solution if

- $r = m$, or
- $r < m$ and $c_i = 0, i = r + 1, \cdots, m$ where c is the right hand side vector after Gauss elimination.

Example

Example

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$$

Solution

Write the equation in schematic representation and proceed with the Gauss elimination

x_1	x_2	x_3	1
1	-1	1	2
2	1	-1	4

x_1	x_2	x_3	1
1	-1	1	2
0	3	-3	0

The rank of this system is $r = 2$ and we have at least one solution, explicitly

$$\begin{aligned} x_2 &= x_3 \\ x_1 &= 2 + x_2 - x_3 = 2 \end{aligned}$$

which means that x_3 is a free parameter. Thus we have infinitely many solutions, $x_1 = 2$, $x_2 = x_3 = \alpha$ where α is any real number.

Example

Example

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$$

Solution

Write the equation in schematic representation and proceed with the Gauss elimination

x_1	x_2	x_3	1
1	-1	1	2
2	1	-1	4

x_1	x_2	x_3	1
1	-1	1	2
0	3	-3	0

The rank of this system is $r = 2$ and we have at least one solution, explicitly

$$\begin{aligned} x_2 &= x_3 \\ x_1 &= 2 + x_2 - x_3 = 2 \end{aligned}$$

which means that x_3 is a free parameter. Thus we have infinitely many solutions, $x_1 = 2$, $x_2 = x_3 = \alpha$ where α is any real number.

Example

Example

$$
\begin{aligned}
x_1 + x_2 &= 2 \\
x_1 - x_2 &= 1 \\
x_1 &= 4
\end{aligned}
$$

Solution

First we write the equation in schematic representation

x_1	x_2	1
1	1	2
1	-1	1
1	0	4

Then we proceed with the Gauss elimination algorithm, giving

x_1	x_2	1
1	1	2
0	-2	-1
0	-1	2

x_1	x_2	1
1	1	2
0	-2	-1
0	0	$\frac{5}{2}$

The rank of this system is $r = 2 < m = 3$ and $c_3 \neq 0$ so that the system has no solution.

Example

Example

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 &= 4 \end{aligned}$$

Solution

First we write the equation in schematic representation

x_1	x_2	1
1	1	2
1	-1	1
1	0	4

Then we proceed with the Gauss elimination algorithm, giving

x_1	x_2	1
1	1	2
0	-2	-1
0	-1	2

x_1	x_2	1
1	1	2
0	-2	-1
0	0	$\frac{5}{2}$

The rank of this system is $r = 2 < m = 3$ and $c_3 \neq 0$ so that the system has no solution.

Outline

Matrix Definitions

Matrix notation is a compact way of expressing systems of linear equations.

A $m \times n$ matrix has m rows and n columns. The entries of a matrix are called **elements**.

The element of a matrix A which lies on the ith row and jth column is denoted by a_{ij} or $(A)_{ij}$. We write a **matrix** as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

For example

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 2 \end{pmatrix}$$

is a 2×3 matrix. In the first row is the second element $(A)_{12} = a_{12} = 3$.

A $n \times n$ matrix has an equal number of rows and columns and is called a **square matrix**.

Two matrices are equal if they are the same size and their elements are the same.

Some Common Matrices

- A $n \times n$ matrix is called a **diagonal matrix** if $(D)_{ij} = 0$ for $i \neq j$. The elements $(D)_{ii} = d_{ii}$ are called the **diagonal elements**. We write

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \text{diag}(5, 2, 3)$$

- The $n \times n$ matrix $I_n = \text{diag}(1, 1, \cdots, 1)$ is called the **identity matrix**. For example

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The 1-column or $n \times 1$ matrices are commonly known as **column vectors** and we write them with lower case letters. The elements of column vectors are called **components**. Components are only identified with a single index. For example, the 4×1 matrix

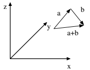
$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 7 \\ 0 \end{pmatrix}$$

is a column vector with $b_1 = 2$, $b_2 = -4$, $b_3 = 7$ and $b_4 = 0$.

Connection with Vectors in Geometry/ Physics (similar but not the same!)

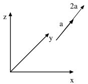
You've probably seen vectors before? e.g. $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, here we usually talk about

components a_1, a_2, a_3 being with respect to a coordinate system e.g. x, y, z
For such vectors we can perform **addition**



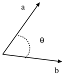
$$\vec{a} + \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

scalar multiplication



$$\alpha\vec{a} = \alpha \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \alpha a_1 \\ \alpha a_2 \\ \alpha a_3 \end{pmatrix}$$

and the **dot product** if $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ is the vector's **length** or **magnitude**



$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{a}||\vec{b}| \cos\theta$$

Same operations apply to column vectors but without reference to a coordinate system.

Addition of matrices

Consider two $m \times n$ matrices A and B. To add the matrices A and B together, we add the respective elements of A and B together. Written more precisely: the $m \times n$ matrix $A + B$ with $(A)_{ij} + (B)_{ij}$ is called the sum of matrices A and B

Example

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Find $A + B$.

Solution

$$A + B = \begin{pmatrix} 3 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix}$$

Addition of matrices

Consider two $m \times n$ matrices A and B. To add the matrices A and B together, we add the respective elements of A and B together. Written more precisely: the $m \times n$ matrix $A + B$ with $(A)_{ij} + (B)_{ij}$ is called the sum of matrices A and B

Example

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

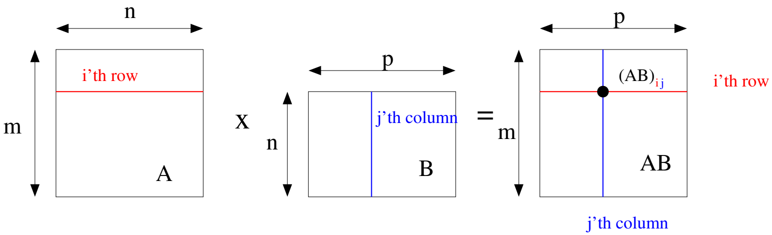
Find $A + B$.

Solution

$$A + B = \begin{pmatrix} 3 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix}$$

Matrix Multiplication

Multiplication by a scalar If a $m \times n$ matrix is multiplied by scaler number α, this means that every element of the matrix is multiplied by α.
Multiplication of two matrices Let A be an $m \times n$ matrix and B a $n \times p$ matrix. The $m \times p$ matrix AB, with $(AB)_{ij} = \sum_{k=1}^{n} (A)_{ik}(B)_{kj}$ is called the matrix product of matrices A and B.
The matrix product AB is only possible if the number of columns of matrix A is exactly the same as the number of rows of matrix B (**due to connection with vector dot product**).



The ith row of matrix A is multiplied by jth of matrix B to obtain the element $(AB)_{ij}$ of matrix AB:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Example

Example

$A = \begin{pmatrix} 3 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$ is a 2×3 matrix and $B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}$ is a 3×4 matrix

Find AB, if possible

Solution

$AB = \begin{pmatrix} 4 & 5 & 2 & 1 \\ 2 & -3 & -5 & 0 \end{pmatrix}$. The two elements in the first column were computed as follows

$$(AB)_{11} = \begin{pmatrix} 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 3 \cdot 1 + 1 \cdot 1 + 0 \cdot 2 = 4$$

$$(AB)_{21} = \begin{pmatrix} 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 2 \cdot 1 + (-2) \cdot 1 + 1 \cdot 2 = 2$$

Example

Example

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$$ is a 2×3 matrix and $$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix}$$ is a 3×4 matrix

Find AB, if possible

Solution

$$AB = \begin{pmatrix} 4 & 5 & 2 & 1 \\ 2 & -3 & -5 & 0 \end{pmatrix}$$. The two elements in the first column were computed as follows

$$(AB)_{11} = \begin{pmatrix} 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 3 \cdot 1 + 1 \cdot 1 + 0 \cdot 2 = 4$$

$$(AB)_{21} = \begin{pmatrix} 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 2 \cdot 1 + (-2) \cdot 1 + 1 \cdot 2 = 2$$

Matrix Multiplication Properties

- For $m \times n$ matrices A and B, the commutative law of addition holds

$$A + B = B + A$$

- For $m \times n$ matrices A, B and C the associative law of addition holds

$$(A + B) + C = A + (B + C)$$

- For every $m \times n$ matrix A, $n \times p$ matrix B and $p \times q$ matrix C, the associative law of multiplication holds

$$(AB)C = A(BC)$$

- For $m \times n$ matrices A and B and $n \times p$ matrices C and D, the distributive law of multiplication holds

$$\begin{aligned} (A + B)C &= AC + BC \\ A(C + D) &= AC + AD \end{aligned}$$

- Note, however that the commutative law of multiplication does NOT hold for matrices. That is to say that in general for two matrices A and B

$$AB \neq BA$$

- For every $m \times n$ matrix A, it holds that $I_m A = AI_n = A$. Thus giving the name for the identity matrix I_m and I_n.

Example

Example

$$A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 4 \\ 5 & 2 \end{pmatrix}$$

Find AB and BA

Solution

$$AB = \begin{pmatrix} 32 & 20 \\ 16 & 10 \end{pmatrix} \neq BA = \begin{pmatrix} 6 & 18 \\ 12 & 36 \end{pmatrix}$$

Example

Example

$$A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 4 \\ 5 & 2 \end{pmatrix}$$

Find AB and BA

Solution

$$AB = \begin{pmatrix} 32 & 20 \\ 16 & 10 \end{pmatrix} \neq BA = \begin{pmatrix} 6 & 18 \\ 12 & 36 \end{pmatrix}$$

Transpose

Let A be a $m \times n$ matrix. Then the $n \times m$ matrix A^T with $(A^T)_{ij} = (A)_{ji}$ is called the transpose of A. A matrix is called symmetric if $A^T = A$ holds.

The matrix transpose obeys the following rules

- For general $m \times n$ matrices A and B, $(A + B)^T = A^T + B^T$ holds
- For every $m \times n$ matrix A and every $n \times p$ matrix B, $(AB)^T = B^T A^T$ holds.

Example

Determine the transpose of the following matrices

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 & -5 \\ 3 & -1 & 2 \\ -5 & 2 & 7 \end{pmatrix}$$

Solution

$$A^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix} \neq A \text{ is NOT symmetric} \qquad B^T = \begin{pmatrix} 2 & 3 & -5 \\ 3 & -1 & 2 \\ -5 & 2 & 7 \end{pmatrix} = B \text{ is symmetric}$$

Transpose

Let A be a $m \times n$ matrix. Then the $n \times m$ matrix A^T with $(A^T)_{ij} = (A)_{ji}$ is called the transpose of A. A matrix is called symmetric if $A^T = A$ holds.

The matrix transpose obeys the following rules

- For general $m \times n$ matrices A and B, $(A+B)^T = A^T + B^T$ holds
- For every $m \times n$ matrix A and every $n \times p$ matrix B, $(AB)^T = B^T A^T$ holds.

Example
Determine the transpose of the following matrices

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 & -5 \\ 3 & -1 & 2 \\ -5 & 2 & 7 \end{pmatrix}$$

Solution

$$A^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix} \neq A \text{ is NOT symmetric} \qquad B^T = \begin{pmatrix} 2 & 3 & -5 \\ 3 & -1 & 2 \\ -5 & 2 & 7 \end{pmatrix} = B \text{ is symmetric}$$

Matrix notation for linear systems

Consider

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Define the matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

and the column vectors

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

The matrix A is called the **coefficient matrix** and b is called the **right hand side** of the linear equation system. The equation system is equivalent to

$$Ax = b$$

This can be solved using the Gauss elimination algorithm

Matrix inverse

The matrix inverse only makes sense for square matrices.

The $n \times n$ matrix X is called the inverse of matrix A if $AX = I_n$.

If the matrix A has an inverse, the matrix A is called **invertible** or **regular**, if the matrix has no inverse it is called **singular**.

For a regular $n \times n$ matrix A, we denote its inverse by A^{-1}.
Let A and B be invertible $n \times n$ matrices, then

- $AA^{-1} = A^{-1}A = I_n$
- A^{-1} is invertible and $(A^{-1})^{-1} = A$
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

The solution to a $n \times n$ linear equation system $Ax = b$ can be computed as $x = A^{-1}b$, since $A^{-1}Ax = I_n x = A^{-1}b$.

But it is much faster to use Gaues elimination!

Calculation of the Matrix Inverse

For a regular $n \times n$ matrix A : Denote the matrix inverse by X and note that

$$AX = \begin{pmatrix} a^{(1)} & \cdots & a^{(n)} \end{pmatrix} \begin{pmatrix} x^{(1)} & \cdots & x^{(n)} \end{pmatrix} = I_n = \begin{pmatrix} b^{(1)} & \cdots & b^{(n)} \end{pmatrix}$$

where $a^{(1)}, \cdots, a^{(n)}$ are the columns of matrix A, $x^{(1)}, \cdots, x^{(n)}$ are the columns of X and $b^{(1)}, \cdots, b^{(n)}$ are the columns of I_n

To determine $x^{(1)}, \cdots, x^{(n)}$, we can solve linear systems $Ax^{(1)} = b^{(1)}, \cdots, Ax^{(n)} = b^{(n)}$ for $x^{(1)}, \cdots, x^{(n)}$. Then, the inverse of A is given $A^{-1} = X$ whose columns are $x^{(1)}, \cdots, x^{(n)}$.

Special case if $n = 2$ then $A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$ if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Example

Example
Determine the inverse of the following matrix

$$A = \begin{pmatrix} 0 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

Solution
We follow a similar procedure to that undertaken when solving linear equations with multiple right hand sides.

x_1	x_2	x_3	1_1	1_2	1_3
0	3	-2	1	0	0
4	-2	1	0	1	0
2	-1	1	0	0	1

x_1	x_2	x_3	1_1	1_2	1_3
2	-1	1	0	0	1
4	-2	1	0	1	0
0	3	-2	1	0	0

x_1	x_2	x_3	1_1	1_2	1_3
2	-1	1	0	0	1
0	0	-1	0	1	-2
0	3	-2	1	0	0

x_1	x_2	x_3	1_1	1_2	1_3
2	-1	1	0	0	1
0	3	-2	1	0	0
0	0	-1	0	1	-2

$$x^{(1)} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ 0 \end{pmatrix},\ x^{(2)} = \begin{pmatrix} \frac{1}{6} \\ -\frac{2}{3} \\ -1 \end{pmatrix},\ x^{(3)} = \begin{pmatrix} \frac{1}{6} \\ \frac{4}{3} \\ 2 \end{pmatrix},\ A^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & -\frac{2}{3} & \frac{4}{3} \\ 0 & -1 & 2 \end{pmatrix}.$$

Note that $A^{-1}A = I_3$

Rank and linear independence

The **rank of a matrix** A is the same as the rank of the linear equation system $Ax = 0$. It is denoted by $r = \text{rank}\, A$.

Linear independence of a set of column vectors means that each of the vectors can **not** be obtained from multiples of the other vectors.

Consider a sequence of n column vectors $a^{(1)}, a^{(2)}, \cdots, a^{(n)}$ each of length m. We can construct a $m \times n$ matrix whose columns are these vectors

$$A = \begin{pmatrix} a^{(1)} & a^{(2)} & \cdots & a^{(n)} \end{pmatrix}$$

We then compute the rank r of the $m \times n$ matrix A :

- If $r = n$ the column vectors are linearly independent.
- If $r < n$ the column vectors are linearly dependent.
- If $r = m$ the column vectors are called generating.
- If $r = n = m$ the column vectors are generating and linearly independent and form a basis.

Example

Example

Determine whether the following vectors are linearly dependent or not

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution

We form the matrix whose columns are the two vectors

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

Next, we perform Gauss elimination on the system $Ax = 0$

x_1	x_2	1_1
1	0	0
1	0	0
1	0	0

x_1	x_2	1_1
1	0	0
0	0	0
0	0	0

We observe that $r = 1$, $m = 3$ and $n = 2$. This means that $r < n$ so that the system is linearly dependent and not generating.

Example

Example
Determine whether the following vectors are linearly dependent or not

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution
We form the matrix whose columns are the two vectors

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

Next, we perform Gauss elimination on the system $Ax = 0$

x_1	x_2	1_1
1	0	0
1	0	0
1	0	0

x_1	x_2	1_1
1	0	0
0	0	0
0	0	0

We observe that $r = 1$, $m = 3$ and $n = 2$. This means that $r < n$ so that the system is linearly dependent and not generating.

Outline

Definition and Properties

Determinates of square matrices can be used to characterise whether a matrix is regular or singular. It can can also be used to compute certain products of vectors (see EG190) and calculate volumes.

A **determinate** is a number which can be computed from each square matrix A.It is written as $\det A$ or $|A|$:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

Example

Determine the determinants of the following matrices

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 4 & 1 & 2 \end{pmatrix}$$

Solution

$$\det A = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 3 \cdot 2 - 2 \cdot 1 = 4$$

$$\det B = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 4 & 1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$$
$$= 1 \cdot 4 - 2 \cdot (-4) + 1 \cdot (-10) = 2$$

Example

Example
Determine the determinants of the following matrices

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 4 & 1 & 2 \end{pmatrix}$$

Solution

$$\det A = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 3 \cdot 2 - 2 \cdot 1 = 4$$

$$\det B = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 4 & 1 & 2 \end{vmatrix} \begin{aligned} &= 1 \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} \\ &= 1 \cdot 4 - 2 \cdot (-4) + 1 \cdot (-10) = 2 \end{aligned}$$

Determinates of larger matrices

We first note the properties

- The determinate of a triangular matrix is equal to the product of the diagonal terms.
- For every $n \times n$ matrix A, it holds that $\det A = \det A^T$
- If the $n \times n$ matrix A is invertible then $\det A \neq 0$ and $\det A^{-1} = \frac{1}{\det A}$

Gauss elimination can be used to reduce a matrix to to upper triangular form. The determinate is then just the product of the diagonal terms!

But, if we swap rows, we must multiply the final result by -1 for each row swap.

Example

Example
Determine the determinant of the following matrix using Gauss elimination

$$A = \begin{pmatrix} 0 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

Solution

$$\det A = \begin{vmatrix} 0 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & -1 & 1 \\ 4 & -2 & 1 \\ 0 & 3 & -2 \end{vmatrix}$$

$$= -\begin{vmatrix} \boxed{2} & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \end{vmatrix}$$

The determinate is then the product of the diagonal terms $\det A = 2 \cdot 3 \cdot (-1) = -6$

Example

Example

Determine the determinant of the following matrix using Gauss elimination

$$A = \begin{pmatrix} 0 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

Solution

$$\det A = \begin{vmatrix} 0 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 1 \end{vmatrix} = -\begin{vmatrix} 2 & -1 & 1 \\ 4 & -2 & 1 \\ 0 & 3 & -2 \end{vmatrix}$$

$$= -\begin{vmatrix} \boxed{2} & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \end{vmatrix}$$

The determinate is then the product of the diagonal terms $\det A = 2 \cdot 3 \cdot (-1) = -6$

Determinates and linear systems

The determinate of a matrix A allow us to say alot about a $n \times n$ linear system. We summarise key results below

- If $\det A \neq 0$ the homogeneous linear equation system $Ax = 0$ has only the trivial solution.
- If $\det A = 0$ the homogeneous linear equation system $Ax = 0$ has infinitely many solutions.
- If $\det A \neq 0$ the linear equation system $Ax = b$ has for a general right hand side vector exactly one solution.
- If $\det A = 0$ the linear equation system $Ax = b$ has no solution or infinitely many solutions, depending on the right hand side vector.

Outline

Eigenvalues and eigenvectors

Consider a $n \times n$ matrix A.

- The number λ is called an **eigenvalue** of matrix A, if there exists a vector (non-trivial) x such that $Ax = \lambda x$ holds.
- If λ is an eigenvalue of the matrix A, then the vector x, for which $Ax = \lambda x$ holds, is called the **eigenvector** of matrix A corresponding to eigenvalue λ.

Noting that λ is an eigenvalue of A when there is a vector $x \neq 0$ such that $Ax - \lambda x = 0$ holds then

$Ax - \lambda I_n x = 0$ or $(A - \lambda I_n)x = 0$.

λ is therefore the eigenvalue of the matrix A when the homogeneous equation system $(A - \lambda I_n)x = 0$ has a non–trivial solution.

It follows that this is exactly the case when $\det(A - \lambda I_n) = 0$.

Example

Example
Determine the eigenvalues of the following matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

Solution

$$A - \lambda I_n = \begin{pmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{pmatrix}$$

Next, we compute the determinate of this matrix

$$\begin{aligned} \det(A - \lambda I) &= (-2-\lambda)\left[(-2-\lambda)^2 - 1\right] - 1\left[(-2-\lambda) - 0\right] \\ &= (-2-\lambda)\left[(-2-\lambda)^2 - 2\right] = -(2+\lambda)(\lambda^2 + 4\lambda + 2) \end{aligned}$$

The cubic equation $\det(A - \lambda I) = 0$ has the following roots, $\lambda_1 = -2$, $\lambda_2 = -2 + \sqrt{2}$ and $\lambda_3 = -2 - \sqrt{2}$ which are also in turn the eigenvalues of matrix A.

Example

Example
Determine the eigenvalues of the following matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

Solution

$$A - \lambda I_n = \begin{pmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{pmatrix}$$

Next, we compute the determinate of this matrix

$$\begin{aligned} \det(A - \lambda I) &= (-2-\lambda)\left[(-2-\lambda)^2 - 1\right] - 1\left[(-2-\lambda) - 0\right] \\ &= (-2-\lambda)\left[(-2-\lambda)^2 - 2\right] = -(2+\lambda)(\lambda^2 + 4\lambda + 2) \end{aligned}$$

The cubic equation $\det(A - \lambda I) = 0$ has the following roots, $\lambda_1 = -2$, $\lambda_2 = -2 + \sqrt{2}$ and $\lambda_3 = -2 - \sqrt{2}$ which are also in turn the eigenvalues of matrix A.

Algebraic multiplicity

In general for a $n \times n$ matrix A we observe that det $(A - \lambda I_n)$ is a polynomial of nth degree in λ.

We call the polynomial det $(A - \lambda I_n)$ the **characteristic polynomial** of matrix A and denote it by $P_A(\lambda)$.

If the polynomial $P_A(\lambda)$ has a root λ^* which is repeated k times, we call k the **algebraic multiplicity** of eigenvalue λ^*.

Example

Example
Determine the eigenvalues of the following matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Solution
This time we use Gauss elimination to compute $\det(A - \lambda I_n)$

$$\det(A - \lambda I_n) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 2-\lambda \\ 1 & 2-\lambda & 1 \\ 2-\lambda & 1 & 1 \end{vmatrix}$$
$$= - \begin{vmatrix} \boxed{1} & 1 & 2-\lambda \\ 0 & 1-\lambda & -1+\lambda \\ 0 & -1+\lambda & 1-(2-\lambda)^2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 2-\lambda \\ 0 & \boxed{1-\lambda} & -1+\lambda \\ 0 & 0 & -4+5\lambda-\lambda^2 \end{vmatrix} \quad (1)$$

Therefore we have $P_A(\lambda) = -(\lambda-1)(4-5\lambda+\lambda^2) = -(\lambda-1)^2(\lambda-4)$. The two eigenvalues are 1 and 4.
The eigenvalue $\lambda = 1$ has algebraic multiplicity 2.
The eigenvalue $\lambda = 4$ has algebraic multiplicity 1.

Example

Example
Determine the eigenvalues of the following matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Solution
This time we use Gauss elimination to compute $\det(A - \lambda I_n)$

$$\begin{aligned} \det(A - \lambda I_n) &= \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2-\lambda \\ 1 & 2-\lambda & 1 \\ 2-\lambda & 1 & 1 \end{vmatrix} \\ &= -\begin{vmatrix} \boxed{1} & 1 & 2-\lambda \\ 0 & 1-\lambda & -1+\lambda \\ 0 & -1+\lambda & 1-(2-\lambda)^2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2-\lambda \\ 0 & \boxed{1-\lambda} & -1+\lambda \\ 0 & 0 & -4+5\lambda-\lambda^2 \end{vmatrix} \end{aligned} \tag{1}$$

Therefore we have $P_A(\lambda) = -(\lambda-1)(4-5\lambda+\lambda^2) = -(\lambda-1)^2(\lambda-4)$. The two eigenvalues are 1 and 4.
The eigenvalue $\lambda = 1$ has algebraic multiplicity 2.
The eigenvalue $\lambda = 4$ has algebraic multiplicity 1.

Eigenvectors

For every eigenvalue, λ, we now wish to compute the non–trivial solution x such that

$$(A - \lambda I_n)x = 0$$

We call this set of nontrivial solutions the **eigenspace of A corresponding to eigenvalue** λ and is given the symbol E_λ. The dimension of E_λ is called the **geometric multiplicity** of the eigenvalue λ. The geometric multiplicity is always greater or equal to 1.
The **span** is the set of all linear combinations of a set of vectors which make up the eigenspace.

Example

Example

Given the following matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

For which we have already found that its eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -2 + \sqrt{2}$ and $\lambda_3 = -2 - \sqrt{2}$, now compute the corresponding eigenspaces

Solution

Eigenspace for $\lambda = -2$. The coefficient matrix in $(A - \lambda I_3)x = 0$ is

$$A + 2I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

With Gauss elimination we can find the eigenspace E_{-2}

x_1	x_2	x_3	1
0	1	0	0
1	0	1	0
0	1	0	0

x_1	x_2	x_3	1
1	0	1	0
0	1	0	0
0	1	0	0

x_1	x_2	x_3	1
1	0	1	0
0	1	0	0
0	0	0	0

The solution set is $\{x_3 = \alpha, x_2 = 0, x_1 = -\alpha | \alpha \in \mathbb{R}\}$ or

Example Continued

$$E_{-2} = \left\{ \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Eigenspace to $\lambda = -2 + \sqrt{2}$. Proceeding with Gauss elimination we have

x_1	x_2	x_3	1
$-\sqrt{2}$	1	0	0
1	$-\sqrt{2}$	1	0
0	1	$-\sqrt{2}$	0

x_1	x_2	x_3	1
1	$-\sqrt{2}$	1	0
$-\sqrt{2}$	1	0	0
0	1	$-\sqrt{2}$	0

x_1	x_2	x_3	1
1	$-\sqrt{2}$	1	0
0	-1	$\sqrt{2}$	0
0	1	$-\sqrt{2}$	0

x_1	x_2	x_3	1
1	$-\sqrt{2}$	1	0
0	-1	$\sqrt{2}$	0
0	0	0	0

The solution has the form $x_3 = \alpha$, $x_2 = \sqrt{2}\alpha$, $x_1 = \alpha$, $\alpha \in \mathbb{R}$. Thus

$$E_{-2+\sqrt{2}} = \left\{ \alpha \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \right\}$$

Example Continued

Eigenspace to $\lambda = -2 - \sqrt{2}$. In a similar fashion to the above, we get

$$E_{-2-\sqrt{2}} = \left\{ \alpha \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \right\}$$