Swansea University
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Engineering Analysis 1 : Integration

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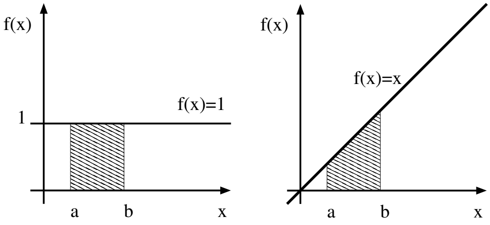
Outline

Basic Ideas and Definitions

As you probably know, the process of finding areas under the graph of a function is called **integration**.

The area under the graph of a function $f(x)$ is called its **integral**.

For simple cases we can work this out from geometry:



Area under graph of $f(x) = 1$ is $1.(b-a) = b-a$

Area under graph of $f(x) = x$ is $a(b-a) + \frac{1}{2}(b-a)^2 = \frac{1}{2}(b^2 - a^2)$,

Why do we need integration as engineers?

Integration has many important applications in engineering, here are just a few:

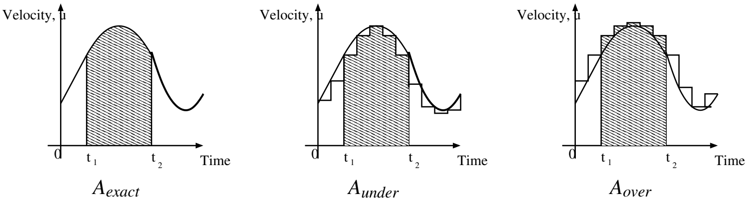
- Calculating the centroid of area;
- Calculating moments of inertia;
- Calculating the work by a variable force;
- The forces due to presence of electrical charges;
- Force exerted by liquid pressure.
- ...

All these applications use the basic techniques we will learn in the coming lectures.

Approximate Integrals

Integration is a process involving **summation**. In fact, we can always approximate an area under a graph by summing rectangles.

Consider a function $u = u(t)$ representing the velocity time graph/ To work out the distance travelled between two times we can approximate the area under the curve.



Clearly the shaded areas satisfy $A_{under} < A_{exact} < A_{over}$.

We can get better estimates of the true area by using **smaller** subdivisions.

In fact, if we were to use infinitely many subdivisions the under and over estimates would both converge to the true area.

Integral definition

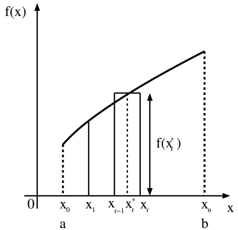
More precisely we define the integral of a function $f(x)$ between $x = a = x_0$ and $x = b = x_n$ as

$$\int_a^b f(x)\mathrm{d}x = \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{r=1}^{n} f(x_r^*)\Delta x_{r-1}$$

where $\Delta x_{r-1} = x_r - x_{r-1}$, $x_{r-1} < x_r^* < x_r$ and $\Delta x = \max_r \Delta x_r$.

But what does this mean?

Understanding the Definition



Definition

$$\int_a^b f(x)\mathrm{d}x = \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{r=1}^{n} f(x_r^*)\Delta x_{r-1}$$

where $\Delta x_{r-1} = x_r - x_{r-1}$, $x_{r-1} < x_r^* < x_r$ and $\Delta x = \max_r \Delta x_r$

The area of each of the rectangles is $f(x_r^*)\Delta x_{r-1}$. If we choose $x_r^* = x_{r-1}$ we underestimate the area, if we choose $x_r^* = x_r$ we over estimate the area.

For a given number of subdivisions n, the summation

$$\sum_{r=1}^{n} f(x_r^*)\Delta x_{r-1}$$

approximates the area under the curve. Taking the limit leads to the exact result for the integral.

Using the Concise Notation

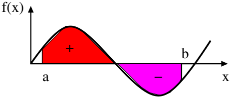
The concise notation for an integral is

$$\int_a^b f(x)\mathrm{d}x$$

where the integration symbol $\int$ is like an elongated S standing for summation.

The $\mathrm{d}x$ is called the **differential** of x, and a and b are called the **limits of integration**.

The function which is being integrated is called the **integrand**.



The contribution to the integral is positive if $f(x) > 0$ and negative if $f(x) < 0$.

The integral is defined as the area **beneath** the curve. For the part where $f(x) < 0$ the contribution to the area is **negative** as the region in magenta is **above** the curve!

Outline

Definite and Indefinite Integrals

A **definite integral** has the form

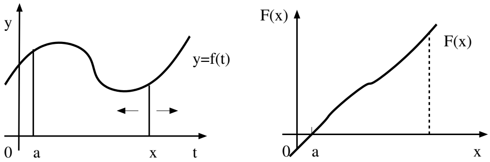
$$\int_a^b f(x)\mathrm{d}x$$

and the result is a **value** for fixed limits.

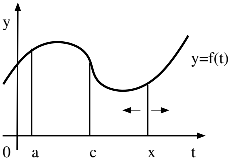
If we replace b by the variable x, the result of the integration is a **function** $F(x)$ that is equal to the area under the graph between a and x

$$F(x) = \int_a^x f(t)\mathrm{d}t$$

we call this type of integral an **indefinite integral**.



Difference Between Two Indefinite Integrals



Consider the indefinite integrals

$$F(x) = \int_a^x f(t)\mathrm{d}t, \qquad G(x) = \int_c^x f(t)\mathrm{d}t$$

with $a < c < x$. Subtracting gives

$$F(x) - G(x) = \int_a^x f(t)\mathrm{d}t - \int_c^x f(t)\mathrm{d}t = \int_a^c f(t)\mathrm{d}t$$

which is an definite integral having constant value equal to the area under the graph between a and c.

Relationship Between a Definite and Indefinite Integral

The most common form of indefinite integral is

$$\int f(x)\mathrm{d}x$$

and without a lower limit the result is always a function plus an **arbitrary constant of integration**, c:

$$\int f(x)\mathrm{d}x = H(x) + c$$

The relationship between a definite and indefinite integral is

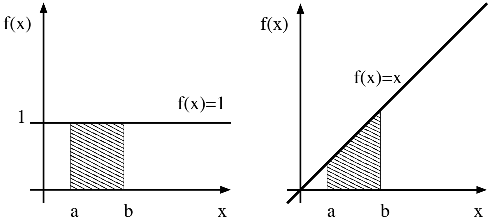
$$\int_a^b f(x)\mathrm{d}x = H(b) - H(a)$$

which is often written as

$$\int_a^b f(x)\mathrm{d}x = [H(x)]_a^b$$

Outline

Simple Integrals



For simple functions such as $f(x) = 1$ and $f(x) = x$ we can compute the function from geometric reasoning.

$$\int_a^b 1 \, \mathrm{d}x = b - a, \qquad \int 1 \, \mathrm{d}x = x + c$$

$$\int_a^b x \, \mathrm{d}x = \frac{1}{2}(b^2 - a^2), \qquad \int x \, \mathrm{d}x = \frac{1}{2}x^2 + c$$

Fundamental Theorem of Calculus

Recall from differentiation that

$$\frac{\mathrm{d}}{\mathrm{d}x}(1) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+k] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + \frac{\mathrm{d}}{\mathrm{d}x}(k) = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)]$$

If we differentiate our two earlier indefinite integrals

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int 1\,\mathrm{d}x\right) = \frac{\mathrm{d}}{\mathrm{d}x}(x+c) = 1$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int x\,\mathrm{d}x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}x^2+c\right) = x$$

This suggests the more general result known as **Fundamental Theorem of Integral and Differential Calculus**

The indefinite integral $F(x)$ *of a continuous function* $f(x)$ *always possesses a derivative* $F'(x)$ *and moreover* $F'(x) = f(x)$

i.e. the process of differentiation is the reverse of that of integration.

Outline

Standard Integrals

$f(x)$	$\int f(x)\,\mathrm{d}x$ Here c is a constant of integration
$x^n \qquad (n \neq -1)$	$\frac{1}{n+1}x^{n+1} + c$
$\frac{1}{x}$	$\left.\begin{array}{ll} \ln x + c & (x > 0) \\ \ln(-x) + c & (x < 0) \end{array}\right\} = \ln\lvert x\rvert + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
e^x	$e^x + c$

By applying the previous definition the above standard indefinite integrals can be obtained.

Example

Question

Check that the fundamental theorem of calculus holds for the integrals of the following

$$f(x) = x^n \qquad n \neq -1 \qquad \text{and } n = -1$$

Solution

We know that

$$\int x^n \mathrm{d}x = \frac{1}{n+1}x^{n+1} + c \qquad \int \frac{1}{x}\mathrm{d}x = \ln|x| + c$$

Applying the fundamental theorem of calculus for the first case gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int x^n \mathrm{d}x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{n+1}x^{n+1} + c\right) = x^n$$

In the second case for $x \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int \frac{1}{x}\mathrm{d}x\right) = \frac{\mathrm{d}}{\mathrm{d}x}(\ln x + c) = \frac{1}{x}$$

and for $x < 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int \frac{1}{x}\mathrm{d}x\right) = \frac{\mathrm{d}}{\mathrm{d}x}(\ln(-x) + c) = \frac{-1}{-x} = \frac{1}{x}$$

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Rules of Integration

- **Rule 1 (scalar–multiplication rule)**
  If k is a constant then
$$\int kf(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x$$
- **Rule 2 (sum rule)**
$$\int [f(x) \pm g(x)]\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$$
- **Rule 3 (linear composite rule)**
  If a and b are constants and $F'(x) = f(x)$ then
$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}F(ax+b) + c$$
- **Rule 4 (inverse–function rule)**
  If $y = f^{-1}(x)$, so that $x = f(y)$, then
$$\int f^{-1}(x)\,\mathrm{d}x = xy - \int f(y)\,\mathrm{d}y$$

Examples

Question
Find the indefinite of $2x^2$
Solution
We use the scalar multiplication rule

$$\int 2x^2 \mathrm{d}x = 2\int x^2 \mathrm{d}x = \frac{2}{3}x^3 + c$$

Example
Determine the indefinite integral of $6x^4 + 4x - \frac{3}{x}$
Solution
Using the sum rule

$$\begin{aligned} \int 6x^4 + 4x - \frac{3}{x}\,\mathrm{d}x &= \int 6x^4\,\mathrm{d}x + \int 4x\,\mathrm{d}x - \int \frac{3}{x}\,\mathrm{d}x \\ &= \frac{6}{5}x^5 + \frac{4}{2}x^2 - 3\ln|x| + c \\ &= \frac{6}{5}x^5 + 2x^2 - 3\ln|x| + c \end{aligned}$$

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Example
Determine the indefinite integral of $\sqrt{5x+2}$
Solution
Using the linear composite rule

$$\int \sqrt{5x+2}\,\mathrm{d}x = \frac{1}{5}\left[\frac{2}{3}(5x+2)^{3/2}\right]+c = \frac{2}{15}(5x+2)^{3/2}+c$$

Example
Determine the indefinite integral of $\ln x$
Solution
If $y=\ln x$ then $x=e^y$ and using the inverse function rule

$$\int \ln x\,\mathrm{d}x = xy-\int e^y\,\mathrm{d}y = xy-e^y+c = x\ln x - x + c$$

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Example

Question
Determine the indefinite integral of $\sin^{-1} x$
Solution
If $y = \sin^{-1} x$ then $x = \sin y$ and using the inverse function rule

$$\begin{aligned} \int \sin^{-1} x \, \mathrm{d}x &= xy - \int \sin y \mathrm{d}y \\ &= xy + \cos y + c \\ &= x \sin^{-1} x + \sqrt{1 - x^2} + c \end{aligned}$$

where the last result follows from the identity $\sin^2 y + \cos^2 y = 1$.

Example

Question
Determine the indefinite integral of $\sin^{-1} x$
Solution
If $y = \sin^{-1} x$ then $x = \sin y$ and using the inverse function rule

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where the last result follows from the identity $\sin^2 y + \cos^2 y = 1$.

Table of Integrals

By applying the standard integrals, basic rules (and occasionally more advanced rules presented later) the table of integrals can be obtained.

$f(x)$	$\int f(x)\,\mathrm{d}x$ Here c is a constant of integration
$x^n \qquad (n \neq -1)$	$\frac{1}{n+1}x^{n+1} + c$
$\frac{1}{x}$	$\left.\begin{array}{ll} \ln x + c & (x > 0) \\ \ln(-x) + c & (x < 0) \end{array}\right\} = \ln\|x\| + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\tan x$	$\ln\|\sec x\| + c$
$\sec x$	$\ln\|\sec x + \tan x\| + c$
$\text{cosec}\, x$	$\ln\left\|\tan \frac{x}{2}\right\| + c$
$\cot x$	$\ln\|\sin x\| + c$
e^x	$e^x + c$
$\ln x$	$x\ln x - x + c$
$\sin^{-1} x$	$x\sin^{-1} x + \sqrt{1-x^2} + c$
$\cos^{-1} x$	$x\cos^{-1} x - \sqrt{1-x^2} + c$
$\tan^{-1} x$	$\frac{1}{2}[2x\tan^{-1} x - \ln(1+x^2)] + c$
$\sinh x$	$\cosh x + c$
$\cosh x$	$\sinh x + c$
$\tanh x$	$\ln(\cosh x) + c$

Outline

Integrals Involving Partial Fractions

To integrate functions of the form

$$\int \frac{p(x)}{q(x)} \mathrm{d}x$$

where $p(x)$ and $q(x)$ are polynomials first expand the rational function in partial fractions.

Example
Using partial fractions, evaluate the indefinite integral of $\frac{6}{x^2-2x-8}$

Solution
We first expand $\frac{6}{x^2-2x-8}$ in terms of partial fractions

$$\frac{6}{x^2-2x-8} = \frac{6}{(x+2)(x-4)} = \frac{-1}{(x+2)} + \frac{1}{x-4}$$

We now evaluate the integral

$$\begin{aligned} \int \frac{6}{x^2-2x-8} \mathrm{d}x &= \int \frac{-1}{(x+2)} \mathrm{d}x + \int \frac{1}{x-4} \mathrm{d}x \\ &= -\ln|x+2| + \ln|x-4| + c \\ &= \ln\left|\frac{x-4}{x+2}\right| + c \end{aligned}$$

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We now evaluate the integral

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Outline

Integration by Parts

We use the product rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \frac{\mathrm{d}u}{\mathrm{d}x}v + \frac{\mathrm{d}v}{\mathrm{d}x}u$$

and rearrange it in the form

$$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - v\frac{\mathrm{d}u}{\mathrm{d}x}$$

and by integrating

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

We can use this result to integrate the product of two functions using a method known as **integration by parts**:

We choose one term to be u and the other to be $\frac{\mathrm{d}v}{\mathrm{d}x}$ and differentiate u to get $\frac{\mathrm{d}u}{\mathrm{d}x}$ and integrate $\frac{\mathrm{d}v}{\mathrm{d}x}$ to get v.

Example

Question
Find the indefinite integral of $x^2 \cos x$
Solution
We first choose $u = x^2$ and $\frac{dv}{dx} = \cos x$ so that $\frac{du}{dx} = 2x$ and $v = \sin x$. Then

$$\begin{aligned} \int x^2 \cos x \, dx &= x^2 \sin x - \int (\sin x)(2x) dx \\ &= x^2 \sin x - \int 2x \sin x dx \end{aligned}$$

The same technique is now repeated for the last integral. Choose $u = 2x$ and $\frac{dv}{dx} = \sin x$ giving $\frac{du}{dx} = 2$ and $v = -\cos x$. Thus

$$\begin{aligned} \int x^2 \cos x \, dx &= x^2 \sin x - \left[(2x)(-\cos x) - \int (-\cos x)(2) \, dx \right] \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + c \end{aligned}$$

Example

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Outline

Integration by Substitution

Composite rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$$

By rearranging and integrating we get

$$\int f'(g(x))g'(x)\,\mathrm{d}x = f(g(x)) + c$$

The key step is to identify the function $g(x)$ that will be **substituted**, note that there can be more than one valid choice and different choices of $g(x)$ may lead to different results for the integral, but will only differ by a constant.

To simplify the process we set the substitution to be $t = g(x)$ so that the integral becomes

$$\begin{aligned}\int f'(g(x))g'(x)\,\mathrm{d}x &= \int f'(t)\frac{\mathrm{d}t}{\mathrm{d}x}\mathrm{d}x = \int f'(t)\,\mathrm{d}t = f(t) + c \\ &= f(g(x))) + c \qquad \text{on back substitution}\end{aligned}$$

This technique is called **integration by substitution**.

Example

Question

Find the indefinite integral of $2x\sqrt{x^2+3}$

Solution

A suitable substitution is $t = x^2+3$, with $\frac{dt}{dx} = 2x$

$$\begin{aligned}\int 2x\sqrt{x^2+3}\,dx &= \int \sqrt{t}\frac{dt}{dx}\,dx = \int \sqrt{t}\,dt \\ &= \frac{2}{3}t^{3/2}+c \\ &= \frac{2}{3}(x^2+3)^{3/2}+c\end{aligned}$$

Example

Find the integral of $\int_0^1 \frac{x+1}{x^2+2x+2}\,dx$

Solution

A suitable substitution is $t = x^2+2x+2$ so that $\frac{dt}{dx} = 2x+2$. However, note the limits of integration change so that when $x=0$, $t=2$ and when $x=1$, $t=5$

$$\begin{aligned}\int_0^1 \frac{x+1}{x^2+2x+2}\,dx &= \frac{1}{2}\int_0^1 \frac{2x+2}{x^2+2x+2}\,dx = \int_0^1 \frac{1}{2t}\frac{dt}{dx}\,dx = \frac{1}{2}\int_2^5 \frac{1}{t}\,dt \\ &= \left[\frac{1}{2}\ln|t|\right]_2^5 = \frac{1}{2}(\ln 5 - \ln 2) = 0.458\ (3dp)\end{aligned}$$

Example

Question

Find the indefinite integral of $2x\sqrt{x^2+3}$

Solution

A suitable substitution is $t = x^2 + 3$, with $\frac{\mathrm{d}t}{\mathrm{d}x} = 2x$

$$\begin{aligned}\int 2x\sqrt{x^2+3}\,\mathrm{d}x &= \int \sqrt{t}\frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \int \sqrt{t}\,\mathrm{d}t \\ &= \frac{2}{3}t^{3/2} + c \\ &= \frac{2}{3}(x^2+3)^{3/2} + c\end{aligned}$$

Example

Find the integral of $\int_0^1 \frac{x+1}{x^2+2x+2}\mathrm{d}x$

Solution

A suitable substitution is $t = x^2 + 2x + 2$ so that $\frac{\mathrm{d}t}{\mathrm{d}x} = 2x + 2$. However, note the limits of integration change so that when $x = 0$, $t = 2$ and when $x = 1$, $t = 5$

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Common Cases

- Integrals of the form

$$\int \frac{g'(x)}{g(x)} \mathrm{d}x$$

where $g'(x)$ is the derivative with respect to x of some function $g(x)$ may be determined by making the substitution $u = g(x)$.

In this case $\frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$ and

$$\int \frac{g'(x)}{g(x)} \mathrm{d}x = \int \frac{\mathrm{d}u}{u} = \ln|u| + c = \ln|g(x)| + c$$

- Another common type is

$$\int g'(x)(g(x))^n \mathrm{d}x \qquad n \neq -1$$

where $g'(x)$ is again the derivative of $g(x)$. If we make the substitution $u = g(x)$ then $\frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$ so that

$$\int g'(x)(g(x))^n \mathrm{d}x = \int u^n \mathrm{d}u = \frac{u^{n+1}}{n+1} + c = \frac{(g(x))^{n+1}}{n+1} + c$$

Examples

Question
Find the integral of $\int \tan ax \mathrm{d}x$
Solution
We write

$$\int \tan ax \mathrm{d}x = \int \frac{\sin ax}{\cos ax} \mathrm{d}x = -\frac{1}{a} \int \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\cos ax)}{\cos ax} = -\frac{1}{a} \ln|\cos ax| + c$$

Question
Find the integral of $\int \frac{(\ln x)^2}{x} \mathrm{d}x$
Solution
We write

$$\int \frac{(\ln x)^2}{x} \mathrm{d}x = \int \frac{1}{x} \cdot (\ln x)^2 \mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}(\ln x)(\ln x)^2 \mathrm{d}x = \frac{(\ln x)^3}{3} + c$$

Examples

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More Complicated Substitutions

If the Integrand contains	Try
$\sqrt{a^2 - x^2}$	$x = a\sin\theta \quad \frac{dx}{d\theta} = a\cos\theta$ or $x = a\tanh u \quad \frac{dx}{du} = a\,\text{sech}^2\,u$
$\sqrt{a^2 + x^2}$	$x = a\sinh u \quad \frac{dx}{du} = a\cosh u$ or $x = a\tan\theta \quad \frac{dx}{d\theta} = a\,\text{sec}^2\,\theta$
$\sqrt{x^2 - a^2}$	$x = a\cosh u \quad \frac{dx}{du} = a\sinh u$ or $x = a\sec\theta \quad \frac{dx}{d\theta} = a\,\text{sec}\,\theta\tan\theta$
Circular Functions	$s = \sin x \quad \frac{ds}{dx} = \cos x$ or $c = \cos x \quad \frac{dc}{dx} = -\sin x$ or $t = \tan\frac{1}{2}x$ $\left(\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \frac{dx}{dt} = \frac{2}{1+t^2}\right)$
Hyperbolic Functions	$u = e^x \quad \frac{du}{dx} = e^x$ or $s = \sinh x \quad \frac{ds}{dx} = \cosh x$ or $c = \cosh x \quad \frac{dc}{dx} = \sinh x$ or $t = \tanh x \quad \frac{dt}{dx} = \frac{1}{2}\,\text{sech}^2\,\frac{1}{2}x$

Example

Question
Find the integral of $\int \sqrt{1-x^2}\mathrm{d}x$
Solution
The integrand contains $\sqrt{a^2-x^2}$ with $a=1$ so we try the substitution $x=\sin\theta$, $\mathrm{d}x/\mathrm{d}\theta=\cos\theta$

$$\int \sqrt{1-x^2}\mathrm{d}x = \int \sqrt{1-\sin^2\theta}\cos\theta\mathrm{d}\theta = \int \cos^2\theta\mathrm{d}\theta$$

Although this simpler than the original integral it is still not immediately integrable. However, by using the identity $\cos 2\theta = 2\cos^2\theta - 1$ gives

$$\int \sqrt{1-x^2}\mathrm{d}x = \int \frac{1}{2}(\cos 2\theta + 1)\mathrm{d}\theta = \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c = \frac{1}{4}\sin(2\sin^{-1}x) + \frac{1}{2}\sin^{-1}x + c$$

Using the identity $\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta}$ we can write this in the alternative form

$$\int \sqrt{1-x^2}\mathrm{d}x = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + c$$

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Outline

Example

Question

Find the integral of $\sin 2x \cos 2x$

Solution

Make the substitution $u = \sin 2x$, $\mathrm{d}u/\mathrm{d}x = 2\cos 2x$ and so

$$\int \sin 2x \cos 2x \mathrm{d}x = \frac{1}{2}\int 2\sin 2x \cos 2x \mathrm{d}x = \frac{1}{2}\int u \mathrm{d}u = \frac{1}{4}u^2 + c$$

Back substituting gives

$$\int \sin 2x \cos 2x \mathrm{d}x = \frac{1}{4}\sin^2 2x + c$$

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Back substituting gives

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Example

Question

Find the integral of $\sin^3 x$

Solution

First write $\sin^3 x = \sin x \sin^2 x$ and then use the identity $\sin^2 x = 1 - \cos^2 x$

$$\int \sin^3 x dx = \int \sin x \sin^2 x dx = \int \sin x dx - \int \sin x \cos^2 dx = -\cos x - \int \sin x \cos^2 x dx$$

For the remaining integral use the substitution $t = \cos x$, $dt/dx = -\sin x$ so

$$\int \sin^3 x dx = -\cos x + \int t^2 dt = -\cos x + \frac{t^3}{3} + c = -\cos x + \frac{\cos^3 x}{3} + c$$

Example

Question
Find the integral of $\sin^3 x$
Solution
First write $\sin^3 x = \sin x \sin^2 x$ and then use the identity $\sin^2 x = 1 - \cos^2 x$

$$\int \sin^3 x\mathrm{d}x = \int \sin x \sin^2 x\mathrm{d}x = \int \sin x\mathrm{d}x - \int \sin x \cos^2 \mathrm{d}x = -\cos x - \int \sin x \cos^2 x\mathrm{d}x$$

For the remaining integral use the substitution $t = \cos x$, $\mathrm{d}t/\mathrm{d}x = -\sin x$ so

$$\int \sin^3 x\mathrm{d}x = -\cos x + \int t^2\mathrm{d}t = -\cos x + \frac{t^3}{3} + c = -\cos x + \frac{\cos^3 x}{3} + c$$

Example

Question
Find the integral of $\sin 4x \cos 2x$
Solution
Apply the identity $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$, obtained by adding $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

Set $A = 4x$ and $B = 2x$ to give

$$\int \sin 4x \cos 2x dx = \frac{1}{2} \int (\sin(2x + 4x) + \sin(4x - 2x)) \, dx = \frac{1}{2} \int (\sin 6x + \sin 2x) \, dx$$

which is integrable and leads to

$$\int \sin 4x \cos 2x dx = -\frac{1}{12} \cos 6x - \frac{1}{4} \cos 2x + c$$

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