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Engineering Analysis 1 : Elementary Functions

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Outline

1. Functions
2. Polynomial Functions
3. Rational Functions
4. Circular Functions
5. Exponential and Logarithmic Functions
6. Continuous and Discontinuous Functions

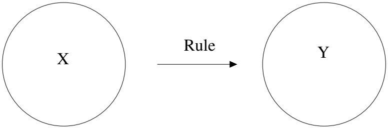
Function Definitions

A function expresses how the **dependent variable** depends on the **independent variable**.

In engineering both variables involved usually have some physical meaning.

A function consists of two different sets of numbers X and Y and rule that assigns to each element x in the set X precisely one element y from the set Y. We denotes this by $f : x \to y$ or $y = f(x)$. We say that x is the **argument** of f.

The set X is called the **domain of the function**. The set Y is called the **codomain of the function**.



Note that there may be elements of Y that are **not** outputs of the function. The set of all images $y = f(x)$, x in X is called the **image set** or **range** of f Hence the range is not always the same as the codomain.

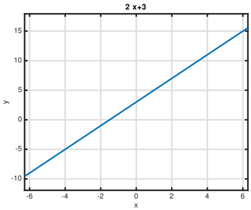
Function Properties

It is worth noting the following points

- In the definition of the function, each input gives rise to exactly one output.
- A function which has the special property that different inputs gives rise to different outputs is said to be **one to one** or **injective**.
- A function which has that has the special property that every element of its codomain is an output (ie whose range is the whole of its codomain) is said to be **on to** or **surjective**.
- A function which is both injective and surjective is called **bijective**.

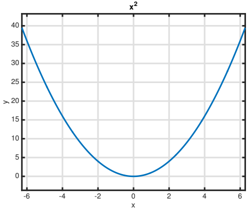
It is important to understand a functions properties in order to be able construct its inverse.

Function Examples



$$y = f(x) = 2x + 3$$

- Domain is $\mathbb{R}$
- Codomain is $\mathbb{R}$
- Range is $\mathbb{R}$
- Function is injective and surjective ie is bijective



$$y = f(x) = x^2$$

- Domain is $\mathbb{R}$
- Codomain is $\mathbb{R}$
- Range is $\mathbb{R}^+ = \{y : y \in \mathbb{R}, y \geq 0\}$
- Function is neither injective or surjective. ie not bijective.

Inverse Functions

For certain $y = f(x)$ it is possible to determine what values of x give rise to a value of y (ie construct its inverse).

In general if a function $y = f(x)$ is bijective (ie injective and surjective)the **inverse function** $x = f^{-1}(y)$ can be constructed (often by simple rearrangement).

Sometimes, if it does not lead to ambiguity, we can interchange x and y and write the inverse function as $f^{-1}(x)$

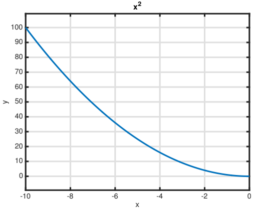
If a function is not bijective, we may need to change the domain and/or codomain to make it bijective.

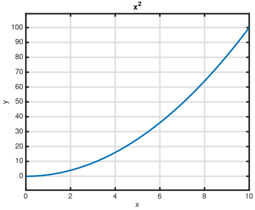
Example

For the bijective function $y = f(x) = ax + b$ with $a \neq 0$ its inverse is $x = f^{-1}(y) = \frac{y-b}{a}$ and we could write $f^{-1}(x) = \frac{x-b}{a}$

Function Examples

$$y = f(x) = x^2$$





- Domain is $\mathbb{R}^-$
- Codomain is $\mathbb{R}^+$
- Range is $\mathbb{R}^+$
- Function is injective and surjective ie is bijective

$$x = f^{-1}(y) = -\sqrt{y}$$

- Domain is $\mathbb{R}^+$
- Codomain is $\mathbb{R}^+$
- Range is $\mathbb{R}^+$
- Function is injective and surjective ie is bijective

$$x = f^{-1}(y) = \sqrt{y}$$

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2 Polynomial Functions

3 Rational Functions

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5 Exponential and Logarithmic Functions

6 Continuous and Discontinuous Functions

Polynomials

A **polynomial function** has the general form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is the a positive integer, a_r is a real number called the **coefficient** of x^r, $r = 0, 1, \cdots, n$.

The index n of the highest power of x occurring in $f(x)$ is called the **degree**.

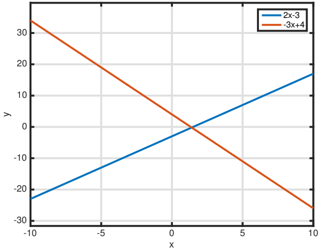
For $n = 1$ we obtain the linear function

$$f(x) = a_1 x + a_0$$

for $n = 2$ we get the quadratic function

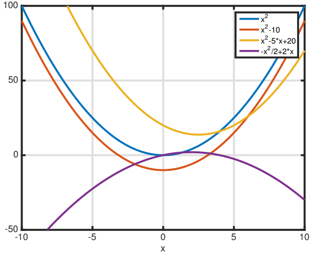
$$f(x) = a_2 x^2 + a_1 x + a_0$$

Graphs of polynomial functions



$y = f(x) = a_1 x + a_0$

- a_0 is the intercept on y axis
- $a_1 = \frac{\Delta f}{\Delta x} = \frac{\Delta y}{\Delta x}$ is the gradient



$y = f(x) = a_2 x^2 + a_1 x + a_0$

- a_0 is the intercept on y axis
- a_1 controls the vertex. By completing the square the vertex is $\left(-\frac{a_1}{2a_2}, -\frac{a_1^2 - 4a_0 a_2}{4a_2}\right)$.
- a_2 controls whether the quadratics opens upward ($a_2 > 0$) or downwards ($a_2 < 0$) and also the speed of increase of the quadratic

Properties of polynomial functions

- If two polynomials are equal for all values of the independent variable, then coefficients of the powers of the variable are equal. Thus if

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\ g(x) &= b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 \end{aligned}$$

  and $f(x) = g(x)$ for all x then $a_i = b_i$ for $i = 0, 1, \cdots, n$.
- Any polynomial with real coefficients can be expressed as a product of **linear and irreducible quadratic factors** (an irreducible quadratic factor is one that cannot be factored into the product of two linear terms with real coefficients).

Example

Question
Find values of A, B and C to ensure that

$$x^2 + 1 = A(x-1) + B(x+2) + C(x^2+2)$$

for all values of x.

Solution
We first gather the appropriate terms together on the right hand side

$$1x^2 + 1 = Cx^2 + (A+B)x + (-A+2B+2C)$$

By comparing and equating coefficients of x^2, x^1 and x^0 we find

$$C = 1 \qquad A + B = 0 \qquad -A + 2B + 2C = 1$$

which lead to the result

$$A = \frac{1}{3} \qquad B = -\frac{1}{3} \qquad C = 1$$

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Example

Question

factorise the polynomial

$$x^3 - 3x^2 + 6x - 4 = 0$$

Solution

The function $f(x) = x^3 - 3x^2 + 6x - 4$ is equal to zero when $x = 1$. We say that $x = 1$ is a **zero** of the function. Thus $x - 1$ must be a factor of $f(x)$. We factor out $x - 1$ as follows

$$x^3 - 3x^2 + 6x - 4 = (x - 1)(x^2 - 2x + 4)$$

To see if we can simplify this further, we need to find if $g(x) = x^2 - 2x + 4$ has linear factors with real coefficients. To check this we check whether $g(x) = 0$ has real roots using

$$x = \frac{2 \pm \sqrt{4 - 4(4)}}{2}$$

In this case we see that are no real roots as $\sqrt{4 - 16} = \sqrt{-12}$ is not defined. Therefore $(x^2 - 2x + 4)$ is an **irreducible quadratic** term and our factorised solution is

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Rational functions

Rational functions have the general form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. If the degree of p is less than the degree of q it is known as **proper** otherwise it is known as **improper**. An improper rational function can always be expressed as a polynomial plus a rational function, e.g.

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

A proper rational function can always be expressed as a sum of simpler functions whose denominator are linear or quadratic irreducible factors, e.g.

$$\frac{x^2+1}{(1+x)(1-x)(2+2x+x^2)} = \frac{1}{1+x} + \frac{1}{5(1-x)} + \frac{4x+7}{5(2+2x+x^2)}$$

these functions are called **partial functions** of the rational function and are often useful in the mathematical analysis of engineering systems.

Method of partial fractions

1. Factorise $q(x)$ fully into linear and irreducible quadratic factors, collecting together all like factors.
2. Each *linear* factor $ax+b$ in $q(x)$ will give rise to a fraction of the type

$$\frac{A}{ax+b}$$

Repeated linear factors $(ax+b)^n$ will give rise to n fractions

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\cdots+\frac{A_n}{(ax+b)^n}$$

Each irreducible quadratic factor ax^2+bx+c in $q(x)$ will give rise to a fraction of the type

$$\frac{Cx+D}{ax^2+bx+c}$$

Sum all fractions together.
3. If the degree of $p(x)$ is n and the degree of $q(x)$ is m and $n \geq m$ then the function is improper and an additional polynomial of the form $B_1+B_2x+\cdots$ of degree $n-m$ must be added.

Method of partial fractions continued

4. Put $\frac{p(x)}{q(x)}$ equal to the sum of all the factors involved.
5. Multiply both sides of the equation by $q(x)$ to obtain an identity involving a polynomial on both the left and right hand side of the equals sign.
6. Compare the coefficients of like powers of x on both sides of the identity. Starting with highest and working towards the lowest power usually makes it easier. The coefficients are then easily found.
7. Check the result by using a test value for x.

Example

Question
Express the following as a partial fraction

$$\frac{3x}{(x-1)(x+2)}$$

Solution
We first write the rational fraction as a sum of simple fractions

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

We then multiply both the top and bottom by $(x-1)(x+2)$ and gather terms in x^1 and x^0.

$$3x = A(x+2) + B(x-1) = (A+B)x + (2A-B)$$

comparing coefficients of powers of x gives

$$3 = A + B \qquad 0 = 2A - B$$

and gives $A = 1$ and $B = 2$ so that

$$\frac{3x}{(x-1)(x+2)} = \frac{1}{(x-1)} + \frac{2}{(x+2)}$$

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Example

Question
Express the following as a partial fraction

$$\frac{3x^2}{(x-1)(x+2)}$$

Solution
In this case the numerator is the same degree as the denominator so we write

$$\frac{3x^2}{(x-1)(x+2)} = A + \frac{B}{(x-1)} + \frac{C}{(x+2)}$$

Multiplying both sides by $(x-1)(x+2)$ and gathering similar terms yields

$$3x^2 = A(x-1)(x+2) + B(x+2) + C(x-1) = Ax^2 + (B+C+A)x + (-2A+2B-C)$$

comparing coefficients of powers of x gives

$$3 = A \qquad 0 = B + C + A \qquad 0 = -2A + 2B - C$$

which results in $A = 3$, $B = 1$ and $C = -4$ and

$$\frac{3x^2}{(x-1)(x+2)} = 3 + \frac{1}{(x-1)} - \frac{4}{(x+2)}$$

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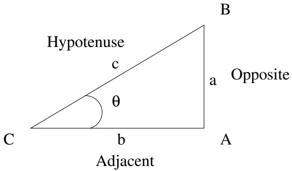
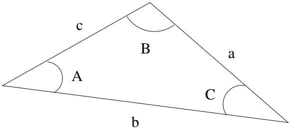
Circular and trigonometric functions

There are two approaches to the definition of **circular or trigonometric functions**. The first **trigonometry** is a static approach is associated with the practical engineering approach of surveying.





Sine and Cosine rules for a general triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Rules for a right angled triangle

$$\sin\theta = \frac{c}{a} = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{b}{a} = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{c}{b} = \frac{\text{opposite}}{\text{adjacent}}$$

Measurement of angles

For problems in trigonometry angles are measured in **degrees**.

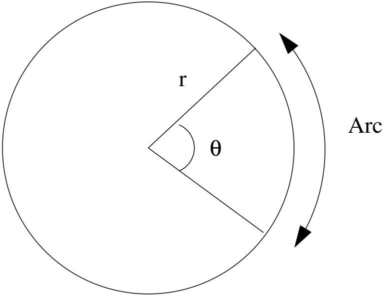
In traditional measurements angles measured in degree are expressed in term of the **sexagesimal system** (degree, minutes and seconds) so that 35.36^o becomes $35^o21'22''$. This measurement system is still commonly employed in Surveying problems in Civil Engineering. On most scientific calculators there is a button which make this conversion straight forward.

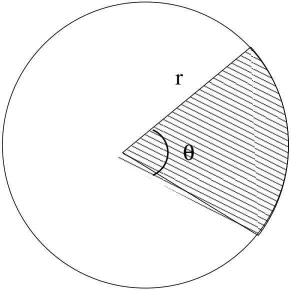
For problems involving circular motion angles are measured in **radians**.

Recall that in 180 degrees there are π radians. This mean that for an angle α measured in degrees is $\pi\alpha/180$ radians and an angle β measured in radians is $180\beta/\pi$ degrees.

Circle properties

For an angle θ measured in radians then





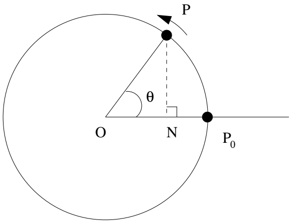
Arc of circle $= r\theta$

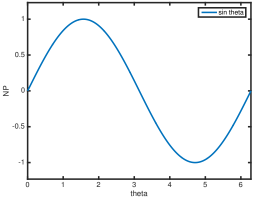
Area of segment $= \frac{1}{2}r^2\theta$

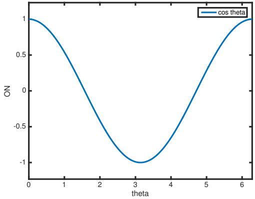
Taking $\theta = 2\pi$ this gives the well known results that the circumference of the circle is $2\pi r$ and its the area is πr^2

Connection with circular motion

Consider a particle P starting at P_0 and moving around a unit circle. The distance $NP = \sin\theta$ and the distance $ON = \cos\theta$.



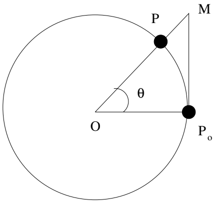


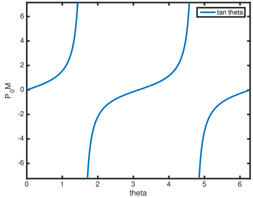


If the circle is of radius r this generalises to
$OP = r\sin\theta$, $ON = r\cos\theta$

Connection with circular motion (continued)

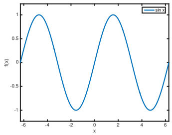
Consider again the particle P starting at P_0 and moving around a unit circle. Extend a line starting at P_0 and at right angles to OP_0 to where the line passing through OP meets this line at point M. The distance $P_0M = \tan\theta$





If the circle is of radius r this generalises to $P_0M = r\tan\theta$

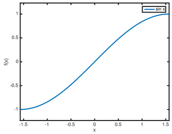
Properties of $f(x) = \sin x$



$f(x) = \sin x$

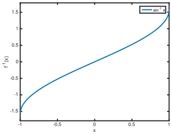
- Periodic with period 2π
- An **odd** function $\sin x = -\sin(-x)$
- Domain is $\mathbb{R}$
- Codomain is $\mathbb{R}$
- Range is $V = \{y \in \mathbb{R} : -1 \leq y \leq 1\}$
- Function is neither injective nor surjective

We must shrink the domain and codomain to allow construction of the inverse function.



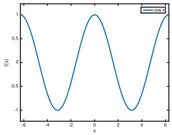
$f(x) = \sin x$

- Domain is $U = \{x \in \mathbb{R} : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$
- Codomain is V
- Range is V
- Function is injective and surjective. Hence is bijective.



$f^{-1}(x) = \sin^{-1} x = \text{arcsin}\, x$

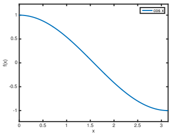
Properties of $f(x) = \cos x$



$f(x) = \cos x$

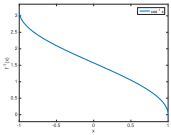
- Periodic with period 2π
- An **even** function $\cos x = \cos(-x)$
- Domain is $\mathbb{R}$
- Codomain is $\mathbb{R}$
- Range is $V = \{y \in \mathbb{R} : -1 \leq y \leq 1\}$
- Function is neither injective nor surjective

We must shrink the domain and codomain to allow construction of the inverse function.



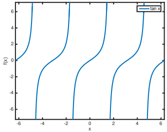
$f(x) = \cos x$

- Domain is $W = \{x \in \mathbb{R} : 0 \leq x \leq \pi\}$
- Codomain is V
- Range is V
- Function is injective and surjective. Hence is bijective.



$f^{-1}(x) = \cos^{-1} x =$ arccos x

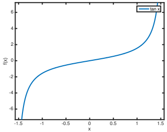
Properties of $f(x) = \tan x$



$f(x) = \tan x$

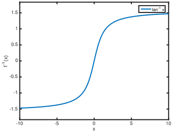
- Periodic with period π
- An **odd** function $\tan x = -\tan(-x)$
- Domain is $X = \{x \in \mathbb{R} : x \neq (2k+1)\frac{\pi}{2} \text{where } k \in \mathbb{Z}\}$
- Codomain is $\mathbb{R}$
- Range is $\mathbb{R}$
- Function is surjective but not injective.

We must shrink the domain to allow construction of the inverse function.



$f(x) = \tan x$

- Domain is $S = \{x \in \mathbb{R} : -\frac{\pi}{2} < x < \frac{\pi}{2}\}$
- Codomain is $\mathbb{R}$
- Range is $\mathbb{R}$
- Function is injective and surjective. Hence is bijective.



$f^{-1}(x) = \tan^{-1} x = \text{arctan}\, x$

Other circular functions

Other circular functions can be defined in terms of the basic functions sine, cosine and tangent.

$$\sec\theta = \frac{1}{\cos\theta} \qquad \text{cosec}\,\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Note that these functions are **different** from the inverse functions and in particular

$$\sec\theta \neq \cos^{-1}\theta \qquad \text{cosec}\theta \neq \sin^{-1}\theta \qquad \cot\theta \neq \tan^{-1}\theta$$

Circular function identities

The triangle identities are

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ 1 + \tan^2 x &= \sec^2 x \\ 1 + \cot^2 x &= \text{cosec}^2\, x \end{aligned}$$

and the compound angle identities are

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \sin(x-y) &= \sin x \cos y - \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \cos(x-y) &= \cos x \cos y + \sin x \sin y \\ \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ \tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x \tan y} \end{aligned}$$

In addition the sum and product identities are easily obtained from the above.

Example

Question
Show that

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Solution
By using $\sin(x+y) = \sin x\cos y + \cos x\sin y$ with $x = y = \theta$ we have

$$\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$$

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Example

Question

Solve the equation

$$2\sin^2 x - 3\sin x + 1 = 0$$

for $0 \leq x \leq 2\pi$

Solution

We first recognise that this is quadratic equation in $\sin x$. Writing $\lambda = \sin x$ we have

$$2\lambda^2 - 3\lambda + 1 = 0$$

The roots of this quadratic equation are given by

$$\lambda = \frac{3 \pm \sqrt{3^2 - 4(2)}}{4} = 1, \ \frac{1}{2}$$

i) When $\lambda = 1$ then $\sin x = 1$ and we have the solution $x = \frac{\pi}{2}$.

ii) When $\lambda = \frac{1}{2}$ then $\sin x = \frac{1}{2}$. Remembering that $\sin x$ is positive in the first and second quadrants we have the solutions $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

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Example

Example

Express $y = 4\sin 3t - 3\cos 3t$ in the form $y = A\sin(3t + \alpha)$

Solution

Use the identity $\sin(x+y) = \sin x \cos y + \sin y \cos x$ and write

$$A\sin(3t+\alpha) = A\sin 3t\cos\alpha + A\sin\alpha\cos 3t$$

We also know that

$$y = 4\sin 3t - 3\cos 3t = A\sin 3t\cos\alpha + A\sin\alpha\cos 3t$$

leading to the two conditions $A\cos\alpha = 4$ and $A\sin\alpha = -3$. Squaring and adding gives

$$A^2\cos^2\alpha + A^2\sin^2\alpha = A^2 = 16 + 9 = 25$$

Thus $A = \pm 5$. Choosing $A = 5$ and dividing the the two previously found conditions gives

$$\frac{A\sin\alpha}{A\cos\alpha} = \tan\alpha = -\frac{3}{4}$$

Now as we have chosen A to be positive, $\cos\alpha$ must be positive and $\sin\alpha$ must be negative and hence α lies in the fourth quadrant. Specifically it is given by $\alpha = -0.64 rad$ and thus

$$y = 5\sin(3t - 0.64)$$

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Outline

1. Functions
2. Polynomial Functions
3. Rational Functions
4. Circular Functions
5. Exponential and Logarithmic Functions
6. Continuous and Discontinuous Functions

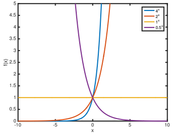
Exponential functions

An **exponential function** has the form

$$f(x) = a^x$$

where a is a positive constant

The graphs of exponential functions are similar:

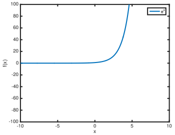


since we can write $y = 4^x = 2^{2x}$ etc. All exponential functions can be written in terms of a single exponential function $y = e^x$ where

$$e = 2.71828128845$$

This number is chosen since the graph $y = e^x$ has the property that the slope of the tangent at any point of the curve is equal to the value of the function at the point.

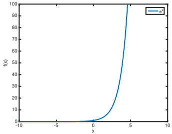
Properties of $f(x) = e^x$



$f(x) = e^x$

- Domain is $\mathbb{R}$
- Codomain is $\mathbb{R}$
- Range is $\mathbb{R}^+$
- Function is injective but not surjective.

We must shrink the codomain to allow construction of the inverse function.



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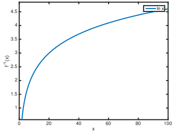


$f^{-1}(x) = \ln x$

General exponential and logarithmic functions

As already stated there are many exponential functions $y = 0.5^x, 2^x, \cdots$ etc and there are also many corresponding logarithmic functions

$$y = a^x \qquad \text{gives } x = \log_a y$$

where $a > 1$ or $0 < a < 1$ is the base of the logarithm.

Note that $\log_{10}$ is frequently referred to as simply $\log x$.

When $a = e$, $\log_a y = \ln y$.

Furthermore, if $y = a^x = e^{kx}$ then by taking natural logarithms then $k = \ln a$

Identities for exponential and logarithmic functions

This exponential function has the following properties

$$
\begin{aligned}
e^{x_1} e^{x_2} &= e^{x_1 + x_2} \\
e^{x+c} &= e^x e^c = A e^x \qquad \text{where } A = e^c \\
\frac{e^{x_1}}{e^{x_2}} &= e^{x_1 - x_2} \\
e^{kx} &= (e^k)^x = a^x \qquad \text{where } a = e^k
\end{aligned}
$$

The logarithmic function has the following properties

$$
\begin{aligned}
\log_a (x_1 x_2) &= \log_a x_1 + \log_a x_2 \\
\log_a \left(\frac{x_1}{x_2} \right) &= \log_a x_1 - \log_a x_2 \\
\log_a x^n &= n \log_a x \\
x &= a^{\log_a x} \\
y^x &= a^{x \log_a y} \\
\log_a x &= \frac{\log_b x}{\log_b a}
\end{aligned}
$$

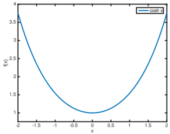
Hyperbolic functions

Associated with the exponential functions is a family of functions called the **hyperbolic functions**. They are defined as follows

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

the reason for these names is geometric. They bear the same relationship to a hyperbola as the circular functions do to the circle.

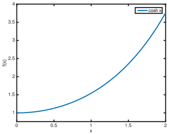
Properties of $f(x) = \cosh x$



$f(x) = \cosh x$

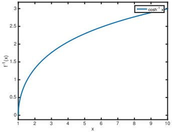
- Domain is $\mathbb{R}$
- Codomain is $\mathbb{R}$
- Range is $V = \{y \in \mathbb{R} : y \geq 1\}$
- Function is neither injective nor surjective.

We must shrink the domain and codomain to allow construction of the inverse function.



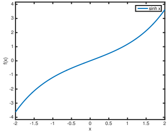
$f(x) = \cosh x$

- Domain is $\mathbb{R}^+$
- Codomain is V
- Range is V
- Function is injective and surjective. Hence is bijective.



$f^{-1}(x) = \cosh^{-1} x = \text{acosh}\, x$

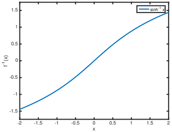
Properties of $f(x) = \sinh x$



$f(x) = \sinh x$

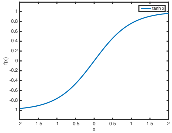
- Domain is $\mathbb{R}$
- Codomain is $\mathbb{R}$
- Range is $\mathbb{R}$
- Function is injective and surjective. Hence is bijective.

We can define the inverse function immediately.



$f^{-1}(x) = \sinh^{-1} x = \text{asinh}\, x$

Properties of $f(x) = \tanh x$



$f(x) = \tanh x$

- Domain is $\mathbb{R}$
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- Range is $Y = \{y \in \mathbb{R} : -1 < y < 1\}$
- Function is injective but not surjective.

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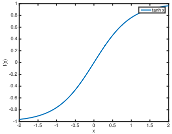

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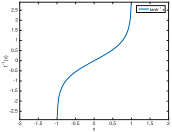

$f^{-1}(x) = \tanh^{-1} x = \text{atanh}\, x$

Other hyperbolic functions and identities

Other hyperbolic functions are defined as

$$\begin{aligned} \text{sech}x &= \frac{1}{\cosh x} \\ \text{cosech}x &= \frac{1}{\sinh x} \qquad (x \neq 0) \\ \text{coth}x &= \frac{1}{\tanh x} \qquad (x \neq 0) \end{aligned}$$

The hyperbolic functions satisfy a series of relationships which include

$$\begin{aligned} \cosh x + \sinh x &= e^x \\ \cosh x - \sinh x &= e^{-x} \\ (\cosh x + \sinh x)(\cosh x - \sinh x) &= e^x e^{-x} \\ \cosh^2 x - \sinh^2 x &= 1 \\ \sinh(x \pm y) &= \sinh x \cosh y \pm \cosh x \sinh y \\ \cosh(x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y \\ \tanh(x \pm y) &= \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \end{aligned}$$

Example

Question
Solve the equation

$$5\cosh x + 3\sinh x = 4$$

Solution
We first express the hyperbolic functions in terms of exponential functions

$$\frac{5}{2}(e^x + e^{-x}) + \frac{3}{2}(e^x - e^{-x}) = 4$$

which gives

$$\begin{aligned} 4e^x - 4 + e^{-x} &= 0 \\ 4(e^x)^2 - 4e^x + 1 &= 0 \end{aligned}$$

where the last equation resulted by multiplying both sides of the equation by e^x. This is quadratic equation in e^x, which we can solve in the standard way

$$e^x = \frac{4 \pm \sqrt{16 - 4(4)}}{8} = \frac{1}{2}$$

Thus $e^x = \frac{1}{2}$ and $x = \ln\frac{1}{2}$

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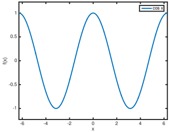
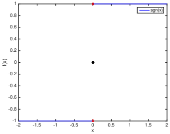
Continuous and discontinuous functions

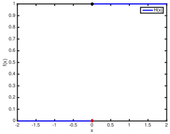
A **continuous function** can be traced from left to right without removing the pen from the paper eg $f(x) = \cos x$



A function which can't be drawn without lifting the pen from paper is called a **discontinuous function**. The jumps in these functions are called **discontinuities**.



sgn x



$H(x)$

Signum and heaviside functions

The **signum function** is a discontinuous function and is defined as

$$\operatorname{sgn} x = \begin{cases} +1 & (x > 0) \\ -1 & (x < 0) \\ 0 & (x = 0) \end{cases}$$

Another discontinuous function is the **Heaviside unit step function**

$$H(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x \geq 0) \end{cases}$$