Swansea University
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Engineering Analysis 1 : Differentiation

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Outline

1. Limits
2. Basic Ideas and Definitions
3. Rules of Differentiation
4. Parametric and Implicit Differentiation
5. Higher Derivatives
6. Optimum Values
7. L'Hôpital's Rule

Introduction to limits

A limit looks at what happens when a function approaches a certain value.

This can be useful if a function can't be evaluated at a point, but instead we can see what happens when we get **closer and closer**.

Example

What happens as x approaches 1 for $f(x) = \dfrac{x^2 - 1}{x - 1}$

Solution

For $x = 1$ then $f(x) = \frac{0}{0}$ is undefined and we say it is **indeterminate**.

But we can investigate what happens as x approaches 1 from below

x	0.5	0.9	0.99	0.999	0.9999	0.99999	$\cdots$
$f(x)$	1.5	1.9	1.99	1.999	1.9999	1.99999	$\cdots$

or from above:

x	1.5	1.1	1.01	1.001	1.0001	1.00001	$\cdots$
$f(x)$	2.5	2.1	2.01	2.001	2.0001	2.00001	$\cdots$

Thus we see that $f(x)$ **approaches** 2 as x approaches 1.

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More formally we say the limit of $f(x)$ as x approaches 1 from below is 2 or:

$$\lim_{x \to 1^-} f(x) = 2$$

and the limit of $f(x)$ as x approaches 1 from above is 2 or:

$$\lim_{x \to 1^+} f(x) = 2$$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$ then we say that **limit** $\lim_{x \to 1} f(x) = 2$ **exists.**

Example

If $f(x) = H(x-1)$ what is $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$? Does $\lim_{x \to 1} f(x)$ exist?

Solution

By considering the definition of the Heaviside unit step function we have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} H(x-1) = \lim_{x \to 0^-} H(x) = 0$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} H(x-1) = \lim_{x \to 0^+} H(x) = 1$$

as $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$ then $\lim_{x \to 1} f(x)$ does not exist in this case.

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Rules for calculating limits

If b and c are constants then

$$\lim_{x \to c} b = b$$

Furthermore if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ and k is a constant

- $\lim_{x \to c} kf(x) = k \cdot \lim_{x \to c} f(x) = kL$
- $\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$
- $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x) = LM$
- $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \dfrac{L}{M}$ if $M \neq 0$

Example

Find the limit of $f(x) = \lim_{x \to 2} \dfrac{4x^3 + 5x + 7}{(x-4)(x+10)}$ if it exists.

Solution

The limit of the numerator is $\lim_{x \to 2} 4x^3 + 5x + 7 = 49$.

The limit of the denominator is $\lim_{x \to 2} (x-4)(x+10) = \lim_{x \to 2} (x-4) \lim_{x \to 2} (x+10) = (2-4)(2+10) = -24 \neq 0$ and so

$$\lim_{x \to 2} \frac{4x^3 + 5x + 7}{(x-4)(x+10)} = -\frac{49}{24}$$

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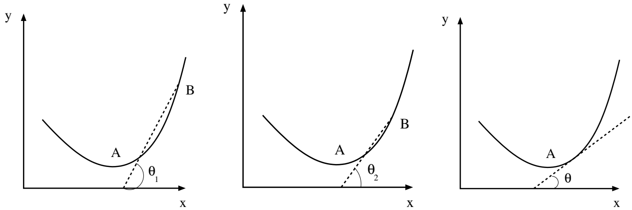
Drawing Tangents



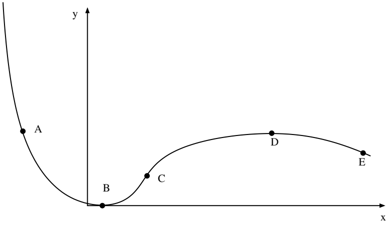
Notice that as the point B moves towards point A then the slope of the **chord** AB tends to the slope of the tangent to the curve at point A.

The **gradient of the curve** at A is the **gradient of the tangent** at that point. The gradient at A is also the **instantaneous rate of change** of the curve at A.

The **gradient** of the curve at the point A is $\tan\theta$ where θ is the angle the tangent line makes with the x axis.

Drawing Tangents

Example
For the curve shown below, state whether the tangent to the curve at points A, B, C, D and E has positive, negative or zero gradient.



Solution
A negative, B zero, C positive, D zero and E negative

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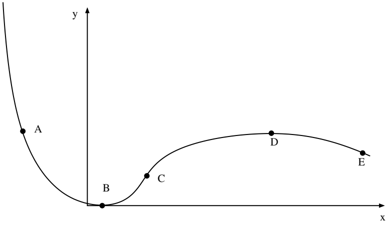


Solution
A negative, B zero, C positive, D zero and E negative

Basic Ideas and Definitions

The (automated) process of finding gradients of tangents to graphs in their solution is called **differentiation** and it measures the rate of change of the value of the functions with respect to its argument.

The gradient of the graph is called the **derivative** of the function.

The derivative of a function $f(x)$ at the point x is formally defined as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

Notation for the Derivative

Two kinds of notation are commonly used for representing the derivative. The first uses a composite symbol

$$\frac{df}{dx} \quad \text{or} \quad df/dx \quad \text{or} \quad D_f$$

The second notation uses a prime

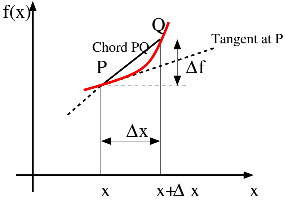
$$f'(x)$$

this means that

$$\frac{df}{dx} = f'(x) = D_f = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Graphical Interpretation

Consider a small change in the dependent variable denoted by Δx. The corresponding change in $f(x)$ is given by Δf



The slope of the line PQ is

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

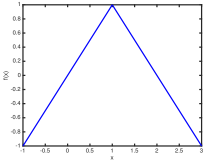
Now in the limit as $\Delta x \to 0$ the point $P \to Q$ and the segment becomes the tangent to the curve at P, whose slope is given by the derivative

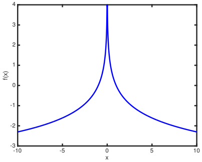
$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

Differentiable Functions

A function $f(x)$ can only be differentiated at points where it has a well defined tangent

Examples of functions with points without well defined tangents are:





The function on the left is **differentiable** at all points except $x = 1$ and the function on right at all points except $x = 0$.

For practical purposes it is sufficient to interpret a differentiable function as one having a smooth continuous graph with no sharp corners. Engineers commonly refer to such functions as being **well–behaved**.

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Standard Derivatives

$f(x)$	$f'(x)$
x^n where n is real	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x

Each of the standard derivatives can be obtained by applying the definition

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(and series expansions, see EG190).

Example

By applying first principles, show that $f'(x) = 2x$ if $f(x) = x^2$.

Solution

By applying the definition we obtain

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + (\Delta x)^2 + 2x\Delta x - x^2}{\Delta x} = 2x$$

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Rules of Differentiation

- **Rule 1 (scaler multiplication rule)**
  If $y = f(x)$ and k is a constant then

$$\frac{\mathrm{d}}{\mathrm{d}x}(ky) = k\frac{\mathrm{d}y}{\mathrm{d}x} = kf'(x)$$

- **Rule 2 (sum rule)**
  If $u = f(x)$ and $v = g(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}x}(u + v) = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x} = f'(x) + g'(x)$$

- **Rule 3 (product rule)**
  If $u = f(x)$ and $v = g(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} = f(x)g'(x) + g(x)f'(x)$$

Rules of Differentiation Continued

- **Rule 4 (quotient rule)**
  If $u = f(x)$ and $v = g(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) - u\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)}{v^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

- **Rule 5 (composite–function or chain rule)**
  If $z = g(x)$ and $y = f(z)$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x} = f'(z)g'(x)$$

- **Rule 6 (inverse–function rule)**
  If $y = f^{-1}(x)$ then $x = f(y)$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y} = \frac{1}{f'(y)}$$

Examples

Question
Find the derivative of $2x^2$
Solution
We use the scalar multiplication rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x^2) = 2\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 4x$$

Question
Find the derivative of $\sin x + x$
Solution
We use the sum rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(x + \sin x) = \frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = 1 + \cos x$$

Question
Find the derivative of $\frac{x^3}{\cos x}$
Solution
We use the quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3}{\cos x}\right) = \frac{\cos x \frac{\mathrm{d}}{\mathrm{d}x}(x^3) - x^3 \frac{\mathrm{d}}{\mathrm{d}x}(\cos x)}{\cos^2 x} = \frac{3x^2 \cos x + x^3 \sin x}{\cos^2 x}$$

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Find the derivative of $\cos\ \sin x$
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We use the chain rule with $z = \sin x$

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Question
Find the derivative of $\ln x$.
Solution
If $y = \ln x$ then $x = e^y$ so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = e^y$$

Then, from the inverse–function rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^y} = \frac{1}{x}$$

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$\sin x$		$\cos x$
$\cos x$		$-\sin x$
$\tan x$		$\sec^2 x$
$\sec x$		$\sec x \tan x$
$\operatorname{cosec} x$		$-\operatorname{cosec} x \cot x$
$\cot x$		$-\operatorname{cosec}^2 x$
e^x		e^x
$\ln x$	$(x \in \mathbb{R}^+)$	$\frac{1}{x}$
$\sin^{-1} x$	$(x \in [-1, 1])$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$(x \in [-1, 1])$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$		$\frac{1}{(1+x^2)}$
$\sinh x$		$\cosh x$
$\cosh x$		$\sinh x$
$\tanh x$		$\frac{1}{\cosh^2 x} = 1 - \tanh^2 x$

More Examples

Example
Show by using only the derivatives for the elementary functions, show that $f'(t) = \sec^2 t$ if $f(t) = \tan t$.
Solution
Note that $f(t) = \dfrac{g(t)}{h(t)}$ with $g(t) = \sin t$ and $h(t) = \cos t$ thus

$$g'(t) = \cos t \qquad h'(t) = -\sin t$$

Then by the quotient rule

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g(t)}{h(t)}\right) = \frac{h(t)g'(t) - g(t)h'(t)}{(h(t))^2} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t$$

where the identity $1 = \cos^t + \sin^2 t$ was applied.

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More Examples

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Show by using only the derivatives for the elementary functions, show that $f'(p) = -\text{cosec}\, p \cot p$ if $f(p) = \text{cosec}\, p$.

Solution
We realise that

$$f(p) = \frac{g(p)}{h(p)} = \frac{1}{\sin p} \qquad g(p) = 1, \qquad h(p) = \sin p$$

so by the quotient rule

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by using the function definitions.

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More Examples

Example
Show by using only the derivatives for the elementary functions, show that $f'(w) = \frac{1}{\sqrt{1-w^2}}$ if $f(w) = \sin^{-1} w$.

Solution
Set $y = f(w) = \sin^{-1} w$ then $w = f^{-1}(y) = siny$. Application of the inverse function rule gives

$$\frac{dw}{dy} = \cos y \qquad \frac{df}{dw} = \frac{dy}{dw} = \frac{1}{dw/dy} = \frac{1}{\cos y}$$

By using the idenitiy $\cos^2 y + \sin^2 y = 1$ it follows that

$$\frac{df}{dw} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

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Parametric Differentiation

A curve may not always be specified in the explicit form $y = f(x)$.

Often curve are expressed **parametrically** as $x = h(t)$ and $y = g(t)$ where t is common independent variable. An example is the curve defining a unit circle $x = \cos t$, $y = \sin t$ where $0 \leq t \leq 2\pi$.

The derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ is obtained from a combination of the chain rule and inverse function:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{g'(t)}{h'(t)}$$

Implicit differentiation

The composite function rule may also be used for differentiating functions expressed in an **implicit form**. For example, let us assume that we have

$$y^3 = x^2$$

To determine the derivative $\frac{dy}{dx}$ we use **implicit differentiation**: Treat y as an unknown function of x and differentiate both sides term by term with respect to x. This gives

$$\frac{d}{dx}(y^3) = \frac{d}{dx}(x^2)$$

Now y^3 is a composite function of x with y being the intermediate variable, so the composite function rule gives

$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\frac{dy}{dx} = 3y^2\frac{dy}{dx}$$

Then substituting back

$$3y^2\frac{dy}{dx} = 2x \qquad \text{or} \qquad \frac{dy}{dx} = \frac{2x}{3y^2}$$

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Higher Derivatives

The derivative $\mathrm{d}f/\mathrm{d}x$ of a function $f(x)$ is itself a function and may be differentiable.

The derivative of a derivative is called the **second derivative** and is written as

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \quad \text{or} \quad f''(x) \quad \text{or} \quad f^{(2)}(x) \quad \text{or} \quad D^2 f$$

To obtain it we simply differentiate $\mathrm{d}f/\mathrm{d}x$ again with respect to x

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

The second derivative may itself be differentiated, yielding **third derivatives** and so on.

In general the n**th derivative** is written as

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} \quad \text{or} \quad f^{(n)}(x) \quad \text{or} \quad D^n f$$

Example

Question
Find the second and third derivatives of $y = \sin 2x$
Solution
We first determine the first derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos 2x$$

We differentiate this to get the second derivative

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = -4\sin 2x$$

Differentiating again yields the third derivative

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = -8\cos 2x$$

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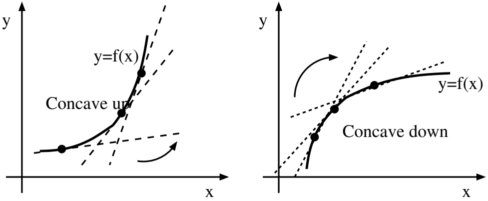
Properties of the Second Derivative

The second derivative $\mathrm{d}^2 f/\mathrm{d}x^2$ represents the rate of change of $\mathrm{d}f/\mathrm{d}x$, this gives us information on how the slope of the tangent changes with increasing x.

- If $\mathrm{d}^2 f/\mathrm{d}x^2 > 0$ then $\mathrm{d}f/\mathrm{d}x$ is increasing as x increases, and the tangent rotates in an anti clockwise direction as we move along the horizontal axis resulting in a graph which is **concave up**.
- If $\mathrm{d}^2 f/\mathrm{d}x^2 < 0$ then $\mathrm{d}f/\mathrm{d}x$ is decreasing as x increases and the tangent rotates in a clockwise direction as move along the horizontal axis resulting in a graph which is **concave down**.



$\mathrm{d}^2 f/\mathrm{d}x^2 > 0$ $\mathrm{d}^2 f/\mathrm{d}x^2 < 0$

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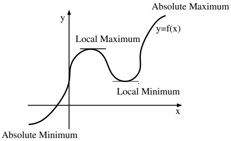
In many situations in engineering we are interested to find **optimum values** eg the **maximum** or **minimum** of a function. This generally occurs when the tangent is horizontal and

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = 0$$

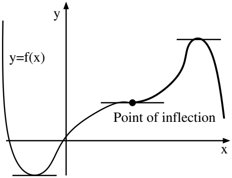


However, the test $f'(x) = 0$ usually leads to points that are usually only **local maxima** and **local minima** of the function. In seeking extremal values we need to also check the end points (if any) of the domain of the function.

Stationary Points

We call the points which have $f'(x) = 0$ **stationary** or **critical points**. As well as possibly being local maxima and local minima these points might also be **points of inflection**.

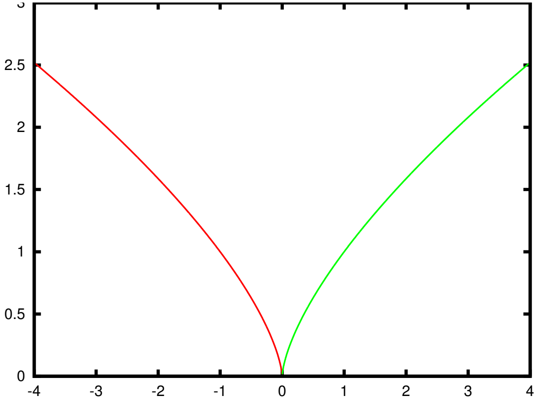
A point of inflection is where the graph crosses its own tangent, which might be be horizontal. In the case where the tangent is horizontal we call it a **stationary point of inflection**.



Other Exceptions

Note that there are also functions which have optimum values at points other than stationary points.

This can happen when the function is not differentiable at a point. A simple example of this is the function $f(x) = x^{2/3}$, which has a minimum at $x = 0$ but as it does not have a unique tangent at this point is not differentiable at this point.



Method 1 for Determining the Nature of Stationary Points

To find local maxima and local minima

- Determine the location of the stationary points where $f'(x) = 0$.
- If the value of $f'(x)$ changes from positive to negative as we pass from left to right through a stationary point then the latter corresponds to a **local maximum**.
- If the value of $f'(x)$ changes from negative to positive as we pass from left to right through a stationary point then the latter corresponds to a **local minimum**.

To find points of inflection

- If $f'(x)$ does not change sign as we pass through a stationary point then the latter corresponds to a point of inflection. In particular we usually call this a **stationary point of inflection** as in this case $f'(x) = 0$ at the point inflection. However points of inflection can occur at other locations too..

Method 2 for Determining the Nature of Stationary Points

To find local maxima and minima

- Find the stationary points $x = a$ where $f'(x) = 0$.
- If $f''(x) < 0$ the function has a **local maximum** at $x = a$.
- If $f''(x) > 0$ the function has a **local minimum** at $x = a$.

To find points of inflection

- Find the points $x = b$ where $f''(b) = 0$
- Check whether $f''(x)$ changes sign as x passes through $x = b$. If this occurs then $x = b$ is a **point of inflection**.
- If in addition $f'(b) = 0$ then $x = b$ is a **stationary point of inflection**.

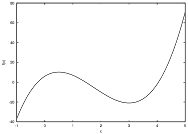
Example

Question

Using method 2 investigate the nature of the stationary points of the function

$$f(x) = 4x^3 - 21x^2 + 18x + 6$$

Solution



The derivative of this function is

$$f'(x) = 12x^2 - 42x + 18$$

Stationary points are where $f'(x) = 0$ and solving the quadratic gives $(0.5, 10.25)$ and $(3, -21)$. The second derivative of the function is

$$f''(x) = 24x - 42$$

At the stationary point $(0.5, 10.25)$, $f''(0.5) = -30 < 0$ and so this corresponds to a local maximum. At the stationary point $(3, -21)$, $f''(3) = 30 > 0$ and so this corresponds to a local minimum. Note that $f''(x) = 0$ at $x = 1.75$ and that $f''(x) < 0$ for $x < 1.75$ and $f''(x) > 0$ for $x > 1.75$. Thus the point $(1.75, -5.375)$ is a point of inflection, but not a stationary point of inflection.

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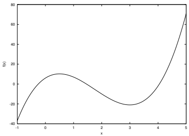

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L'H$\hat{o}$pital's Rule

L'H$\hat{o}$pital's rule allows us to compute limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

in the case where $f(a) = g(a) = 0$. In this case

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

provided that $g'(a) \neq 0$.

It may be that $\frac{f'(a)}{g'(a)}$ is also indeterminate. In such cases we must repeat the process of applying L'H$\hat{o}$pital's rule each time we have $\frac{0}{0}$ at $x = a$. We **must stop** through if one or more of the derivatives is non-zero.

Applications of L'Hôpital's Rule

Example

Using L'Hôpital's Rule determine the limit

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$

Solution

Since $\frac{\sin x - x}{x^3}$ is indeterminate at $x = 0$ we apply L'Hôpital's Rule

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \qquad \text{again } 0/0 \text{ at } x = 0$$

$$= \lim_{x\to 0} \frac{-\sin x}{6x} \qquad \text{again } 0/0 \text{ at } x = 0$$

$$= \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}$$

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Applications of L'Hôpital's Rule

Example
Using L'Hôpital's Rule determine the limit

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$$

Solution
Since $\frac{1-\cos x}{x+x^2}$ is indeterminate at $x=0$ we apply L'Hôpital's Rule

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}=0$$

Note in this case the limit is zero as $\frac{\sin x}{1+2x}$ has the form of $0/1$ at $x=0$. If we mistakingly proceeded we would get

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

which is incorrect.

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