Engineering Analysis 2 : Ordinary Differential Equations

P. Rees, O. Kryvchenkova and P.D. Ledger,

engmaths@swansea.ac.uk

College of Engineering, Swansea University, UK

Outline

1 Differential Equations

2 Classification of Differential Equations

3 First Order ODEs

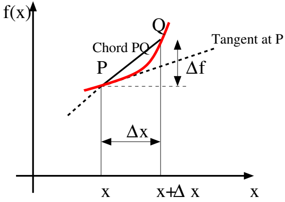
4 Special Types

5 Second Order ODEs

Differential Equations -1

Differential equations and **differentiation** are *not* the same thing, but are closely related.

Recall that **differentiation** is the process of obtain equations defining tangents to curves. The derivative tells us the rate of change of a function.



and is obtained as the limit $\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$.

A **differential equation** is an equation which contains derivatives and the goal is usually to solve it. ie To find the function (for engineers usually a **physical field** (see vectors)) that satisfies the equation. This process will usually involve some form of **integration**.

Differential Equations -2

Why are differential equations important?

Differential equation govern nearly all physical processes, which engineers are interested in. Here are some examples:

- **Maxwell equations** are important to electrical and medical engineers, they describe electromagnetisim and optics. Simplified versions include Kirchhoff's law and Ohm's law.
- **Heat Transfer**, and the Laws of Thermodynamics are important for mechanical engineers and describe how heat and energy are exchanged between different bodies.
- **Navier Stokes Equations** describe fluid flow (including both liquids and gasses) and are important for mechanical, medical, aerospace and civil engineers. Simplified versions include Darcy's law, Bernoulli's equation, and mass conservation.
- **Diffusion equations** are important to Chemical and Medical engineers. They describe how chemical species evolve over time. Simplified versions include Fick's law.
- **Navier's Equation for Elasticity** is important for describing mechanical displacements and stresses and strains. It is important for aerospace, civil and mechanical engineers.

Before we can attempt to solve any differentiation equations, we first need to classify them as the classification will determine the approach used to solve them.

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2. Classification of Differential Equations
3. First Order ODEs
4. Special Types
5. Second Order ODEs

Ordinary and Partial Differential Equations -1

We are already familiar with the rules of ordinary differentiation.

eg. If $f(x) = x^2$ then $\dfrac{\mathrm{d}f}{\mathrm{d}x} = 2x$

Associated with ordinary differentiation are **ordinary differential equations** or **ODE's** for short. These are equations containing ordinary derivatives.

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 2x \qquad \text{and} \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}f}{\mathrm{d}x} = \cos 2x$$

are examples of ODEs.

In engineering ODEs arise when a physical process (described by the previously named equations) can simplified.

ie the **field**, in this case f, is only the function of one spatial or temporal variable, in this case x.

Ordinary and Partial Differential Equations -2

More generally, physical processes occur in multiple dimensions (eg x, y, z) and possibly also evolve over time.

In such cases we need to use different types of differentiation called **partial differentiation**. We'll be learning more about this shortly. We distinguish it from normal differentiation using curl d's (ie ∂ rather d),

Associated with partial differentiation are **partial differential equations** or **PDE's** for short. These are equations containing partial derivatives.

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 2x^2 + 4y$$

is an example of a PDE.

Each of the previously named differential equations are PDEs in their general form. In the above the **field**, f, is a function of both x and y.

In this section will focus on **ODEs** rather than PDEs.

Independent and Dependent Variables

The variables to which differentiation occurs are called the **independent variables**. These will usually be coordinates (eg x, y, z) or time (t).

The variable being differentiated are called the **dependent variables**. These will usually be the **physical fields** (which possibly depend on x, y, z, t).

- An ODE usually has 1 dependent and 1 independent variables.
- A PDE has 2 or more independent variables

In the ODE

$$\frac{d^2f}{dx^2} + 2x\frac{df}{dx} = \sin 2x$$

the independent variable is x and the dependent variable is f. The two ODEs

$$\begin{aligned} \frac{du}{dt} + 2\frac{dv}{dt} - 2u + 3v &= \cosh t \\ 2\frac{du}{dt} + 3\frac{dv}{dt} + 5u + 2v &= \sinh t \end{aligned}$$

are coupled and the independent variable is t and the dependent variables are u and v.

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Order of a Differential Equation

To further classify a differential equation we often talk about its order. The **order of a differential equation** is the degree of the highest derivative in the differential equation. Thus

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}f}{\mathrm{d}x} = \sin 2x$$

is a **second order ODE**. The coupled ODEs

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} - 2x + 3y &= \cosh t\\ 2\frac{\mathrm{d}x}{\mathrm{d}t} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 5x + 2y &= \sinh t\end{aligned}$$

are both **first order**. Also the differential equation

$$\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 4\frac{\mathrm{d}x}{\mathrm{d}y} = 0$$

is **first order** despite the term $\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2$.

Linear and Non-linear Equations

Linear differential equations are those in which the dependent variable (or variables) and their derivatives do not occur as products, raised to powers or in non–linear functions. **Nonlinear equations** are those which are not linear.
The coupled equations

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} + 2\frac{\mathrm{d}v}{\mathrm{d}t} - 2u + 3v &= \cosh t \\ 2\frac{\mathrm{d}u}{\mathrm{d}t} + 3\frac{\mathrm{d}v}{\mathrm{d}t} + 5u + 2v &= \sinh t \end{aligned}$$

are examples of linear differential equations. Whereas

$$\begin{aligned} \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + 4\frac{\mathrm{d}u}{\mathrm{d}y} &= 0 \\ \frac{\mathrm{d}^2u}{\mathrm{d}t^2} + u\frac{\mathrm{d}u}{\mathrm{d}t} &= 4\sin t \\ 4\frac{\mathrm{d}u}{\mathrm{d}t} + \sin u &= 0 \end{aligned}$$

are all non–linear differential equations.

Homogeneous and Non–homogeneous Equations

We always arrange an ODE so that any terms involving the dependent variable appear on the left hand side of the equation and any just involving the independent variable appear on the right. If the right hand side is zero we call it a **homogeneous ODE**.

If the right handside is non-zero it is called a **non–homogeneous ODE**.

The equations

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} + 4f &= 0 \\ 4\frac{\mathrm{d}f}{\mathrm{d}t} + f\sin t &= 0 \end{aligned}$$

are homogeneous differential equations. The equations

$$\begin{aligned} \frac{\mathrm{d}^2u}{\mathrm{d}t^2} + t\frac{\mathrm{d}u}{\mathrm{d}t} &= 4\sin t \\ \frac{\mathrm{d}^2u}{\mathrm{d}x^2} - 4u\frac{\mathrm{d}u}{\mathrm{d}x} &= \cos 2x \end{aligned}$$

are non–homogeneous differential equations.

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Implicit and Explicit Solutions-1

We consider first order ODE's that can be put in the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} = g(u, x)$$

here g is any function of u and x.

- If we can find a solution of the form $u(x) = \cdots$ we call this an **explicit solution** to the ODE.
- If it is not possible to arrange the solution in the form where the dependent variable can be expressed only as function of the independent variable we have instead an **implicit solution** in the form $f(u, x) = 0$.

Implicit and Explicit Solutions-2

Example

Show that $x + y + e^{xu} = 0$ is an implicit solution to the differential equation

$$(1 + xe^{xu})\frac{\mathrm{d}u}{\mathrm{d}x} + 1 + ue^{xu} = 0$$

Solution

Differentiating $x + u + e^{xu} = 0$ with respect to x gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(x + u + e^{xu}) &= 0 \\
1 + \frac{\mathrm{d}u}{\mathrm{d}x} + e^{xu}\left(u + x\frac{\mathrm{d}u}{\mathrm{d}x}\right) &= 0 \\
(1 + ue^{xu})\frac{\mathrm{d}y}{\mathrm{d}x} + 1 + ue^{xu} &= 0
\end{aligned}$$

and so $x + u + e^{xu} = 0$ is an implicit solution to the differential equation.

General and Particular Solutions

The implicit solution we have just seen has no arbitrary constants and we call this type of solution a **particular solution**. But in the previous case, we did not obtain the solution, we merely checked it agreed with a given ODE.

We shortly see solution process to ODE leads to solutions with arbitrary constants.

For example, the differential equations $\frac{\mathrm{d}u}{\mathrm{d}x} = u$ has infinitely many solutions of $u = Ae^{x}$ where A is any real constant.

We say that this is the **general solution of the differential solution**. The general solution of a first order differential equation has one arbitrary constant.

To determine the constant we need to supply either boundary or initial conditions.

Boundary and Initial Conditions

Initial conditions usually relate to prescribing a value of the dependent variable u (or its derivatives) when the independent variable is zero. They make most sense when the independent variable is time, t.

Physically, initial conditions describe the behaviour of an engineering system at time $t = 0$, eg at rest.

Boundary conditions relate to prescribing a value of the dependent variable u (or its derivatives) for a given value of the independent variable x.

Physically, boundary conditions describe known behaviour of an engineering system at certain locations x, eg known zero displacement of a simply supported beam..

A differential equation together with its boundary conditions is called a **boundary value problem**. A differential equation together with its initial conditions is called a **initial value problem**.

Example
The differential equation $\frac{du}{dx} = u$ has the general solution $u = Ae^x$. Work out the particular solution for the case when $u(0) = u(x = 0) = 1$.
Solution
By substituting $u(0) = Ae^0 = 1$ we find that $A = 1$ and thus have the solution $u = e^x$

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Variable Separable Type

ODE's of the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} = k(u)\ell(x)$$

are called **variable separable type** differential equations. This means that $g(u,x)$ can be written as $g(u,x) = k(u)\ell(x)$, ie a function of u times a function of x. Note that not every function can be written in this way (eg $g(u,x) = 1 + xu$).

To solve variable separable type differential equations, assuming that $k(u) \neq 0$ we write

$$\frac{\mathrm{d}u}{k(u)} = \ell(x)\mathrm{d}x$$

so that the terms on the right hand side of the equation are involving y and those on the left just involve x. Next we integrate to get the general solution.

$$\int \frac{\mathrm{d}u}{g(u)} = \int \ell(x)\mathrm{d}x + A$$

Integration Reminder

Before showing examples further let us recall the following important rules from EG189 that we will frequently use

- $\int \frac{\mathrm{d}x}{x} = \ln|x| + C$
- From the linear composite rule we have

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

- Special case 1 of integration by substitution

$$\int g^n(x)g'(x)\mathrm{d}x = \frac{1}{n+1}g^{n+1}(x) + C \qquad n \neq -1$$

- Special case 2 of integration by substitution

$$\int \frac{g'(x)}{g(x)}\mathrm{d}x = \ln|g(x)| + C$$

- Integration by parts

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

Variable Separable Type

Example
Determine the general solution to the differential equation $\frac{\mathrm{d}u}{\mathrm{d}x} = u$
Solution
We first assume $u \neq 0$ and write $\frac{\mathrm{d}u}{u} = \mathrm{d}x$ and integrating we get

$$\begin{aligned} \int \frac{\mathrm{d}u}{u} &= \int \mathrm{d}x \\ \ln|u| &= x + B \\ |y| &= e^{x+B} = Ce^{x} \end{aligned}$$

where C is a positive constant. Now $|u|$ could mean either u or $-u$ so that

$$\begin{aligned} u &= \pm Ce^{x} \\ u &= Ae^{x} \end{aligned}$$

where A is a nonzero constant. But, $u = 0$ is also a solution and so $u = De^{x}$ with D any real constant.

Variable Separable Type

Example

Determine the general solution to the differential equation $\frac{du}{dx} = \frac{u-1}{x+3}$. Find the particular solution for which $u(0) = -1$

Solution

We first assume $u - 1 \neq 0$ and write $\frac{du}{u-1} = \frac{dx}{x+3}$ integrating gives

$$\begin{aligned} \int \frac{du}{u-1} &= \int \frac{dx}{x+3} \\ \ln|u-1| &= \ln|x+3| + A \\ |u-1| &= e^{\ln|x+3|+A} = e^A|x+3| \\ |u-1| &= B|x+3| \end{aligned}$$

where B is a positive constant. We remember that $|X| = |U|$ implies that $X = U$ or $X = -U$ so that the general solution is

$$\begin{aligned} u - 1 &= \pm B(x+3) \\ u - 1 &= C(x+3) \end{aligned}$$

where C is nonzero constant. The particular solution for $u(0) = -1$ gives $C = -\frac{2}{3}$ and $u - 1 = -\frac{2}{3}(x+3)$

Separable after Substitution Type

Some first order ODEs are not directly separable but become separable after making a simple substitution. First order ODEs that can be put in the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} = k\left(\frac{u}{x}\right)$$

where $k(\cdot)$ is a function of a single variable is differential equation of this type.
We put $v = \frac{u}{x}$ where v is a function of x and obtain an ODE that is satisfied by v and x.
To see this, we set $u = vx$ then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

by the product rule and thus we can write

$$\frac{\mathrm{d}u}{\mathrm{d}x} = k\left(\frac{u}{x}\right) = k(v) = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

so that we have

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{k(v) - v}{x}$$

which is of the separable type with general solution

$$\int \frac{\mathrm{d}v}{k(v) - v} = \int \frac{\mathrm{d}x}{x} + A$$

After integrating we replace v by u/x.

Separable after Substitution Type

Example

Find the general solution to the first order ODE

$$3xu^2\frac{\mathrm{d}u}{\mathrm{d}x} = x^3 + u^3$$

Solution

We observe that $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x^3+u^3}{3xu^2}$ is not of the separable type. If we divide the top and bottom by x^3 we get

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1+(u/x)^3}{3(u/x)^2} = k(u/x) \qquad \text{where} k(t) = \frac{1+t^3}{3t^2}$$

This is the separable after substitution type so we let $v = u/x$ or $y = ux$. Thus

$$\frac{\mathrm{d}u}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+v^3}{3v^2}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+v^3}{3v^2} - v = \frac{1-2v^3}{3v^2}$$

Separable after Substitution Type

and we have separable differential equation $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}\frac{1-2v^3}{3v^2}$ and if we that assume $\frac{1-2v^3}{3v^2} \neq 0$ we get

$$\begin{aligned} \int \frac{3v^2}{1-2v^3}\mathrm{d}v &= \int \frac{\mathrm{d}x}{x} \\ -\frac{1}{2}\ln|1-2v^3| &= \ln|x| + A \\ -A &= \ln(|x|\sqrt{|1-2v^3|}) \\ e^{-A} = B &= |x|\sqrt{|1-2v^3|} \end{aligned}$$

where B is any real non–zero constant, inserting $v = u/x$ and squaring gives

$$x^2\frac{|x^3-2u^3|}{|x^3|} = B^2 \qquad |x^3-2u^3| = B^2|x|$$

Separable after Substitution Type

Thus we have

$$(x^3 - 2u^3) = \pm B^2 x \qquad (x^3 - 2u^3) = Cx$$

where C is any non zero constant. If we substitute $C = 0$ we get

$$x^3 - 2u^3 = 0 \qquad x^3 = 2u^3 \qquad \frac{1}{2} = \left(\frac{u}{x}\right)^3 = v^3$$

thus $v^3 = 1/2$. In turns out that this is indeed a solution to the differential equation $\frac{dv}{dx}$ and is exactly the solution for which $\frac{1-2v^3}{3v^2} = 0$. Thus the general solution is

$$u = \left(\frac{x^3 - Cx}{2}\right)^{1/3}$$

with C any real number.

Linear Type

Most general first order **linear type** ODE's are of the form

$$R(x)\frac{\mathrm{d}u}{\mathrm{d}x} + S(x)u = T(x)$$

where $R(x)$, $S(x)$ and $T(x)$ are given functions of x. Note that if $T(x) = 0$ then the ODE is of the separable type already discussed. For cases when $T(x) \neq 0$ then we put the equation into **standard form** by dividing by $R(x)$ to get

$$\frac{\mathrm{d}u}{\mathrm{d}x} + N(x)u = M(x)$$

where $N(x) = S(x)/R(x)$ and $M(x) = T(x)/R(x)$.

To solve this type of ODE we multiply the equation by $e^{\int N(x)\mathrm{d}x}$. This is called the **integrating factor** of the ODE and gives

$$e^{\int N(x)\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + uN(x)e^{\int N(x)\mathrm{d}x} = M(x)e^{\int N(x)\mathrm{d}x}$$

Linear Type

Now

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\int N(x)\mathrm{d}x}u\right) &= e^{\int N(x)\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\int N(x)\mathrm{d}x}\right)\\ &= e^{\int N(x)\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} + yN(x)e^{\int N(x)\mathrm{d}x}\end{aligned}$$

So that we have the ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\int N(x)\mathrm{d}x}u\right) = M(x)e^{\int N(x)\mathrm{d}x}$$

When we integrate this equation we get

$$e^{\int N(x)\mathrm{d}x}u = \int\left(M(x)e^{\int N(x)\mathrm{d}x}\right)\mathrm{d}x + A$$

Hence the general solution to the equation is

$$u = e^{-\int N(x)\mathrm{d}x}\left[\int\left(M(x)e^{\int N(x)\mathrm{d}x}\right)\mathrm{d}x\right] + Ae^{-\int N(x)\mathrm{d}x}$$

Linear Type

Example
Find the general solution to the ODE

$$x\frac{\mathrm{d}u}{\mathrm{d}x} - u = \frac{x^4}{\sqrt{1+x^3}} \qquad \text{with } x > 0$$

and find the particular solution that satisfies $u(2) = 6$.
Solution
We first write the equation in standard form

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{x}u = \frac{x^3}{\sqrt{1+x^3}}$$

In this case $N(x) = -1/x$ and the integrating factor is

$$e^{-\int \frac{1}{x}\mathrm{d}x} = e^{-\ln|x|} = e^{\ln|x|^{-1}} = e^{\ln\frac{1}{|x|}} = \frac{1}{|x|} = \frac{1}{x}$$

since $x > 0$.

Linear Type

If we multiply the differential equation by the integrating factor we get

$$\begin{aligned}\frac{1}{x}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{x^2}u &= \frac{x^2}{\sqrt{1+x^3}}\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}u\right) &= \frac{x^2}{\sqrt{1+x^3}}\end{aligned}$$

If we integrate both sides we get

$$\begin{aligned}\frac{1}{x}u &= \int \frac{x^2}{\sqrt{1+x^3}}\mathrm{d}x + A\\ \frac{1}{x}u &= \frac{2}{3}\sqrt{1+x^3} + A\\ u &= \frac{2}{3}x\sqrt{1+x^3} + Ax\end{aligned}$$

where A is any real constant.

Linear Type

The particular solution for which $u(2) = 6$ gives

$$6 = \frac{2}{3} \cdot 2 \cdot \sqrt{9} + 2A \qquad \text{hence } A = 1$$

Thus

$$u = x\left(\frac{2}{3}\sqrt{1+x^3} + 1\right)$$

Outline

1. Differential Equations
2. Classification of Differential Equations
3. First Order ODEs
4. Special Types
5. Second Order ODEs

Special Types

In addition to the separable, separable after substitution and linear type first order ODEs. We can also solve the following types of ODEs:

- **Type A.** ODE's of the type
$$\frac{\mathrm{d}u}{\mathrm{d}x} = g(ax + bu)$$
where a and b are known constants and g is a known function. Can be solved by a substitution.
- **Type B.** ODE's of the type
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{ax + bu + e}{cx + fu + g}$$
is not separable but we can make a simple substitution to make it the same as in type A or separable after a substitution so that we can solve it.
- **Type C.** ODE's of the type
$$\frac{\mathrm{d}u}{\mathrm{d}x} + P(x)u = Q(x)u^n$$
when $n \neq 0$. The is also known as **Bernoulli's equation**. We note that when $n = 0$ then this is a first order linear equation and when $n = 1$ then the equation is a first order linear separable equation. In general we divide through by u^n and do a change of variables.

Details of the solution of these more specialised types can be found in the accompanying lecture notes and will not be covered in the lectures.

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Type C

Example

Find the general solution to the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}x} - 5u = -\frac{5}{2}xu^3$$

Solution

We recognise it as an example of the Bernoulli equation. Assuming that $u \neq 0$ and dividing by u^3 we get

$$u^{-3}\frac{\mathrm{d}u}{\mathrm{d}x} - 5u^{-2} = -\frac{5}{2}x$$

If we let $z = u^{-2}$ then $\frac{\mathrm{d}z}{\mathrm{d}x} = -2u^{-3}\frac{\mathrm{d}u}{\mathrm{d}x}$ then the ODE becomes

$$-\frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}x} - 5z = -\frac{5}{2}x$$

which when expressed in standard form is

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 10z = 5x$$

Type C

The integrating factor is $e^{\int 10\mathrm{d}x} = e^{10x}$. Multiplying by the integrating factor gives

$$\begin{aligned}
e^{10x}\frac{\mathrm{d}z}{\mathrm{d}x} + 10ze^{10x} &= 5xe^{10x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{10x}z\right) &= 5xe^{10x} \\
e^{10x}z &= 5\int xe^{10x}\mathrm{d}x = 5\left[\frac{xe^{10x}}{10} - \int \frac{e^{10x}}{10}\mathrm{d}x\right] \\
&= 5\left[\frac{xe^{10x}}{10} - \frac{e^{10x}}{100}\right] + A
\end{aligned}$$

where A is an arbitrary constant of integration. We have that

$$z = \frac{x}{2} - \frac{1}{20} + Ae^{-10x} \qquad \text{so finally} \qquad u = \left(\frac{x}{2} - \frac{1}{20} + Ae^{-10x}\right)^{-1/2}$$

Earlier we made the assumption that $u \neq 0$ however, $u = 0$ also satisfies the ODE so it is also a solution.

Outline

1. Differential Equations
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Second Order ODEs - 1

For simplicity we shall only consider second order linear ODE's. These are equations of the form

$$\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = g(x) \tag{1}$$

here $P(x)$, $Q(x)$ and $g(x)$ are all given continuous functions.

For the special case where $g(x) = 0$ then

$$\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u = 0$$

and this is called a second order **homogeneous differential equation**.

Second Order ODEs - 2

The **general solution to second order homogeneous equations** is

$$u(x) = A_1 u_1(x) + A_2 u_2(x)$$

where A_1 and A_2 are arbitrary constants and $u(x) = u_1(x)$ and $u(x) = u_2(x)$ are **both** solutions of

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}u}{\mathrm{d}x} + Q(x)u = 0$$

and are linearly independent i.e. $u_1(x)$ is not just a multiple of $u_2(x)$.

The **general solution to second order non–homogeneous equations** is

$u(x) =$ general solution of the homogeneous equation $+$ any particular solution to (1)

We call the general solution to the homogeneous equation the **complementary function** and the particular solution the **particular integral**.

Linear Equations with Constant Coefficients

We further restrict consideration to where $P(x)$ and $Q(x)$ are just constants. ie equations of the type

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}u}{\mathrm{d}x} + a_0u = g(x)$$

where a_1 and a_0 are constants and $g(x)$ is continuous.

There are three steps to the solution:

1. We first obtain the solution we need to find the complementary function u_{cf} by solving the homogeneous problem (ie assuming $g(x) = 0$).
2. If $g(x) \neq 0$ we need to also find u_{pi}, the particular integral, otherwise $u_{pi} = 0$.
3. The solution is $u(x) = u_{cf} + u_{pi}$

To Find the Complementary Function -1

This is the solution to the differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}u}{\mathrm{d}x} + a_0u = 0$$

to find the solution to this equation, we first write down the polynomial equation

$$m^2 + a_1m + a_0 = 0$$

this is an equation in m and is called the **auxiliary equation**.

This equation has roots

$$m_1, m_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

The roots will be either be real and distinct, real repeated or complex.

To Find the Complementary Function-2

Depending on the roots, the complementary function is

- If m_1 and m_2 are real and distinct then the the complementary function is of the form

$$u_{cf} = A_1 e^{m_1 x} + A_2 e^{m_2 x}$$

  where A_1, A_2 are arbitrary constants

- If the roots are equal $m_1 = m_2$ then the complementary function is given by

$$u_{cf} = (A_1 + A_2 x) e^{m_1 x}$$

  where A_1, A_2 are arbitrary constants

- If the roots are complex then m_1 and m_2 are complex conjugates say $p \pm jq$ then the complementary function is

$$u_{cf} = e^{px}(A_1 \cos qx + A_2 \sin qx)$$

  where A_1, A_2 are arbitrary constants

To Find the Complementary Function -3

Example

Find the general solution to the ODE

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + 2\frac{\mathrm{d}u}{\mathrm{d}t} + 10u = 0$$

Solution

We first write down the auxiliary equation

$$m^2 + 2m + 10 = 0$$

which has roots

$$m = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm j3$$

As the ODE is homogeneous the general solution is $u = u_{cf} = e^{-t}(A_1 \cos 3t + A_2 \sin 3t)$ and A_1, A_2 are real constants.

To Find the Particular Integral-1

This is some particular solution $u(x) = u_{pi}(x)$ that solves the equation

$$\frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_0 u = g(x)$$

We always find the complementary function u_{cf} first.

Then, to find u_{pi}, we use the trial method. We make a guess for u_{pi} depending on the form of $g(x)$ and u_{cf}. Then we substitute our guess into the ODE.

- Suppose that $g(x) = Ae^{kx}$ where A and k are given constants.
  - If k is not a root of the auxiliary equation, try $u_{pi}(x) = ae^{kx}$
  - If k is a simple root of the auxiliary equation, try $u_{pi}(x) = axe^{kx}$
  - If k is a double root of the auxiliary equation, try $u_{pi}(x) = ax^2e^{kx}$

To Find the Particular Integral-2

Example

Find the general solution to

$$\frac{\mathrm{d}^2u}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}x} = 5e^x$$

Solution

To find the complementary function, we first find the roots of the auxiliary equation $m^2 - m = m(m-1) = 0$. Hence the roots are $m = 0$ and $m = 1$. Thus the complementary function is

$$u_{cf} = A_1e^{0x} + A_2e^{1x} = A_1 + A_2e^x$$

where A_1 and A_2 are arbitrary constants. Since $m = 1$ is a simple root of the auxiliary equation we try $u_{pi} = axe^x$. Thus

$$\frac{\mathrm{d}u_{pi}}{\mathrm{d}x} = ae^x(1+x) \qquad \frac{\mathrm{d}^2u_{pi}}{\mathrm{d}x^2} = ae^x(2+x)$$

To Find the Particular Integral-3

and so u_{pi} is a solution provided that

$$\begin{aligned} ae^x(2+x) - ae^x(1+x) &= 5e^x \\ ae^x &= 5e^x \\ a &= 5 \end{aligned}$$

Thus $u(x) = u_{pi}(x) = 5xe^x$ is a particular solution to $\frac{\mathrm{d}^2u}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}x} = 5e^x$. The general solution is

$$u(x) = u_{cf}(x) + u_{pi}(x) = A_1 + A_2e^x + 5xe^x$$

To Find the Particular Integral-4

- Suppose that $g(x) = p_0 + p_1 x + \cdots + p_k x^k$ where $p_0, p_1, \cdots, p_k$ are given constants
  - If 0 is not a root of the auxiliary equation try $u_{pi}(x) = b_0 + b_1 x + \cdots + b_k x^k$
  - If 0 is a simple root of the auxiliary equation try $u_{pi}(x) = x(b_0 + b_1 x + \cdots + b_k x^k)$
  - If 0 is a double root of the auxiliary equation try $u_{pi}(x) = x^2(b_0 + b_1 x + \cdots + b_k x^k)$

  In all cases substitute $u(x) = u_{pi}(x)$ into $\frac{\mathrm{d}^2 u}{\mathrm{d}x} + a_1 \frac{\mathrm{d}u}{\mathrm{d}x} + a_0 u = g(x)$ and determine the constants $b_0, b_1, \cdots, b_k$.

To Find the Particular Integral-5

Example

Find the general solution to

$$\frac{\mathrm{d}^2u}{\mathrm{d}x} - 2u = x^2 + 2$$

Solution

For the complementary function the auxiliary equation is $m^2 - 2 = 0$ and has roots $m = \pm\sqrt{2}$. Thus the complementary function is

$$u_{cf} = A_1 e^{\sqrt{2}x} + A_2 e^{-\sqrt{2}x}$$

For the particular integral we try $u_{pi} = b_0 + b_1 x + b_2 x^2$ where b_0, b_1, b_2 are constants to be found. Differentiating we have

$$\frac{\mathrm{d}u_{pi}}{\mathrm{d}x} = b_1 + 2b_2 x \qquad \frac{\mathrm{d}^2u_{pi}}{\mathrm{d}x^2} = 2b_2$$

To Find the Particular Integral-6

and substituting this into the differential equation gives

$$2b_2 - 2(b_0 + b_1x + b_2x^2) = x^2 + 2$$

Equating coefficients of x^2 gives $-2b_2 = 1$ so that $b_2 = -1/2$. Equating coefficients of x gives $-2b_1 = 0$ and it follows that $b_1 = 0$. Finally equating coefficients of x^0 gives $2b_2 - 2b_0 = 2$ which gives $b_0 = -3/2$. Thus the particular integral is

$$u_{pi} = -\frac{3}{2} - \frac{1}{2}x^2$$

and the general solution is

$$u(x) = u_{cf}(x) + u_{pi}(x) = A_1e^{\sqrt{2}x} + A_2e^{-\sqrt{2}x} - \frac{3}{2} - \frac{1}{2}x^2$$

To Find the Particular Integral-7

- Suppose that $g(x) = A \sin kx + B \cos kx$ where A, B and k are given constants.
  - If $\sin kx$ is not a term in the complementary function, try $u_{pi} = a \cos kx + b \sin kx$
  - If $\sin kx$ is a term in the complementary function try $u_{pi} = x(a \cos kx + b \sin kx)$

  As before, subsitute $u(x) = u_{pi}(x)$ into $\frac{d^2u}{dx} + a_1 \frac{du}{dx} + a_0 u = g(x)$ and determine the constants a, b.

- Suppose that $g(x) = g_1(x) + g_2(x)$ where u_1 is a solution of $\frac{d^2u}{dx} + a_1 \frac{du}{dx} + a_0 u = g_1(x)$ and u_2 is a solution of $\frac{d^2u}{dx} + a_1 \frac{du}{dx} + a_0 u = g_2(x)$. Then $u(x) = u_1(x) + u_2(x)$ is a solution of $\frac{d^2u}{dx} + a_1 \frac{du}{dx} + a_0 u = g(x)$.