Engineering Analysis 2 : Complex Numbers

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Outline

The Number j

Recall that $a^2 \geq 0$ for any **real** number a and that square root of a negative **real** number is not defined as a real number.

In this part of the course we shall introduce a new set of numbers that allow us to make sense of numbers such as $\sqrt{-9}$.

In particular we introduce a new number, j, for which

$$j^2 = -1 \qquad \text{so that} \qquad j = \sqrt{-1}$$

j is not real and is instead an **imaginary** number. The symbol i is sometimes used in place of j.

We can now make sense of $\sqrt{-9} = \sqrt{-1}\sqrt{9} = j3$

The Complex Number $z = a + jb$

Recall that from EG189 that the general roots of $a_2x^2 + a_1x + a_0 = 0$ are given by

$$x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

This result gives rise to the following implications

- For $a_1^2 > 4a_2a_0$ we have two real roots
- For $a_1^2 = 4a_2a_0$ we have one repeated root
- For $a_1^2 < 4a_2a_0$ we have no real roots.

We can now make sense of the case $a_1^2 < 4a_2a_0$ in terms of j. We will see in the next slide that the result are two **complex numbers** each expressed in **Cartesian form**

$$z = a + jb$$

where $Re(z) = a$ is called the real part of z and $Im(z) = b$ is called the imaginary part of z. The set of all complex numbers is $\mathbb{C}$.

Example

Question
Determine the roots of the quadratic equation

$$x^2 - 6x + 10 = 0$$

Solution
To find the roots we simply apply the equation

$$x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

with $a_2 = 1$, $a_1 = -6$ and $a_0 = 10$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(10)}}{2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm j = 3 + j \text{ or } 3 - j$$

Example

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Determine the roots of the quadratic equation

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$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(10)}}{2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm j = 3 + j \text{ or } 3 - j$$

Outline

Addition, Subtraction and Multiplication

To add or subtract two complex numbers we simply perform the operations on their respective real and imaginary parts.

For example, if $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ then

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

and

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

For the multiplication of complex numbers we make use of the fact that $j^2 = -1$

$$\begin{aligned} z_1 z_2 = (x_1 + jy_1) \times (x_2 + jy_2) &= x_1 x_2 + jx_1 y_2 + jx_2 y_1 + j^2 y_1 y_2 \\ &= x_1 x_2 - y_1 y_2 + j(x_1 y_2 + x_2 y_1) \end{aligned}$$

Example

Question
Determine $z_1 \times z_2$ where $z_1 = 3 + j2$ and $z_2 = 5 + j3$
Solution

$$\begin{aligned} z_1 z_2 &= z_1 \times z_2 = (3 + j2) \times (5 + j3) \\ &= 15 + j9 + j10 + j^2 6 \\ &= 15 + j19 - 6 \\ &= 9 + j19 \end{aligned}$$

Example

Question

Determine $z_1 \times z_2$ where $z_1 = 3 + j2$ and $z_2 = 5 + j3$

Solution

$$
\begin{aligned}
z_1 z_2 &= z_1 \times z_2 = (3 + j2) \times (5 + j3) \\
&= 15 + j9 + j10 + j^2 6 \\
&= 15 + j19 - 6 \\
&= 9 + j19
\end{aligned}
$$

Division

The devision of complex numbers slightly more complicated, consider the complex number

$$z = \frac{x_1 + jy_1}{x_2 + jy_2}$$

to evaluate this expression we multiply the top and bottom by $x_2 - jy_2$ giving

$$\begin{aligned} z &= \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} \\ &= \frac{x_1x_2 + y_1y_2 + j(x_2y_1 - x_1y_2)}{(x_2^2 + y_2^2)} \end{aligned}$$

the number $x - jy$ is called the **complex conjugate** of $z = x + jy$ and is denoted by z^*.

Properties of the Complex Conjugate

Given a complex number $z = x + jy$ its complex conjugate is $z^* = x - jy$

The complex conjugate has the following properties

- $z + z^* = 2x = 2\text{Re}(z)$
- $z - z^* = j2y = j2\text{Im}(z)$
- $zz^* = x^2 + y^2 = |z|^2$

where $|z|$ is the **modulus** of z, which is a positive real number.

Note also that complex roots of a quadratic equation always appear in **complex conjugate pairs** ie $x + jy$ and $x - jy$

Example

Question

Determine $\frac{z_1}{z_2}$ where $z_1 = 1 + j4$ and $z_2 = 3 - j2$. Henceforth determine the real and imaginary parts of $\frac{z_1}{z_2}$ and $\left|\frac{z_1}{z_2}\right|$

Solution

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{1+j4}{3-j2} = \frac{(1+j4)(3+j2)}{(3-j2)(3+j2)} \\ &= \frac{3+j2+j12+j^2 8}{9+j6-j6-j^2 4} \\ &= \frac{-5+j14}{13}\end{aligned}$$

The real part of $\frac{z_1}{z_2}$ is $-\frac{5}{13}$ and the imaginary part is $\frac{14}{13}$. The modulus of $\frac{z_1}{z_2}$ is $\sqrt{\frac{221}{169}}$.

Example

Question
Determine $\frac{z_1}{z_2}$ where $z_1 = 1 + j4$ and $z_2 = 3 - j2$. Henceforth determine the real and imaginary parts of $\frac{z_1}{z_2}$ and $\left|\frac{z_1}{z_2}\right|$

Solution

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{1+j4}{3-j2} = \frac{(1+j4)(3+j2)}{(3-j2)(3+j2)} \\ &= \frac{3+j2+j12+j^2 8}{9+j6-j6-j^2 4} \\ &= \frac{-5+j14}{13} \end{aligned}$$

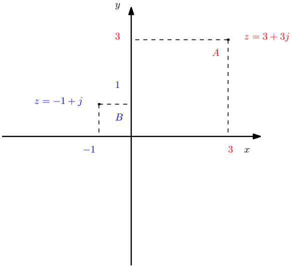
The real part of $\frac{z_1}{z_2}$ is $-\frac{5}{13}$ and the imaginary part is $\frac{14}{13}$. The modulus of $\frac{z_1}{z_2}$ is $\sqrt{\frac{221}{169}}$.

Outline

Agrand Diagram

Complex numbers in Cartesian form $z = x + jy$ can be displayed graphically on an **Argand diagram**. This resembles a graph with each individual complex number being a point with coordinates (x, y).

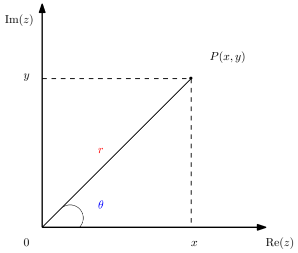
The x-axis is Re(z) and the y-axis is Im(z).



The point A with coordinates $(3, 3)$ corresponds to $z = 3 + j3$ and the point B with coordinates $(-1, 1)$ corresponds to $z = -1 + j$.

Agrand Diagram

The Argand diagram also makes it possible to introduce an alternative representation of a complex number.



The length of the line OP is the **modulus** of the complex number

$$r = |z| = \sqrt{x^2 + y^2}$$

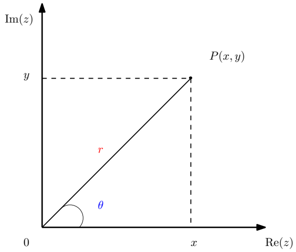
The angle that the line OP makes with the positive x real axis is the **argument** of the complex number

$$\theta = \arg z = \tan^{-1}(y/x)$$

Note that the polar coordinates (r, θ) and $(r, \theta + 2\pi)$ represent the same point on the Argand diagram.

Outline

Polar Form of a Complex Number



The Argand diagram makes it clear that

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

It therefore follows that the complex number $z = x + jy$ can be expressed in the form

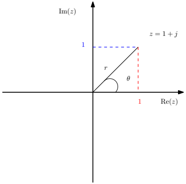
$$z = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta)$$

This is called the **polar form** of the complex number and is frequently written as $r\angle\theta$

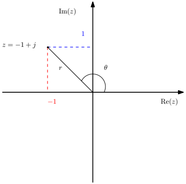
$$z = r\angle\theta = r(\cos\theta + j\sin\theta)$$

Computing the Argument

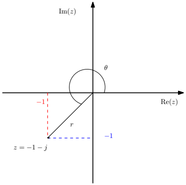
Great **care** is required when applying $\theta = \arg z = \tan^{-1}(y/x)$ on a calculator!



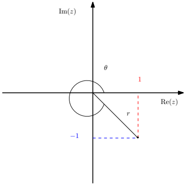
$$z = 1 + j = 1 + 1j$$
$$|z| = \sqrt{2},\ \theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$$



$$z = -1 + j = -1 + 1j$$
$$|z| = \sqrt{2},\ \theta = \pi + \tan^{-1}\frac{1}{-1} = \frac{3\pi}{4}$$



$$z = -1 - j = -1 + (-1)j$$
$$|z| = \sqrt{2},\ \theta = \pi + \tan^{-1}\frac{-1}{-1} = \frac{5\pi}{4}, (-\frac{3\pi}{4})$$



$$z = 1 - j = 1 + (-1)j$$
$$|z| = \sqrt{2},\ \theta = 2\pi + \tan^{-1}\frac{-1}{1} = \frac{7\pi}{4}, (-\frac{\pi}{4})$$

Example

Question

For $z_1 = 2 + j3$ and $z_2 = 3 - j2$ determine their polar form

Solution

For z_1 we have

$$\begin{aligned} r_1 = |z_1| &= \sqrt{2^2 + 3^2} = \sqrt{13} \\ \theta_1 = \arg z_1 &= \tan^{-1}\frac{3}{2} = 0.982(3dp) \end{aligned}$$

Thus $z_1 = 2 + j3 = \sqrt{13}(\cos 0.982 + j\sin 0.982)$. For z_2 we have

$$\begin{aligned} r_2 = |z_2| &= \sqrt{3^2 + (-2)^2} = \sqrt{13} \\ \theta_2 = \arg z_2 &= -\tan^{-1}\frac{2}{3} = -0.588(3dp) \end{aligned}$$

This means that

$z_2 = 3 - j2 = \sqrt{13}(\cos(-0.588) + j\sin(-0.588)) = \sqrt{13}(\cos 0.588 - j\sin 0.588)$

Example

Question
For $z_1 = 2 + j3$ and $z_2 = 3 - j2$ determine their polar form
Solution
For z_1 we have

$$\begin{aligned} r_1 = |z_1| &= \sqrt{2^2 + 3^2} = \sqrt{13} \\ \theta_1 = \arg z_1 &= \tan^{-1}\frac{3}{2} = 0.982(3dp) \end{aligned}$$

Thus $z_1 = 2 + j3 = \sqrt{13}(\cos 0.982 + j\sin 0.982)$. For z_2 we have

$$\begin{aligned} r_2 = |z_2| &= \sqrt{3^2 + (-2)^2} = \sqrt{13} \\ \theta_2 = \arg z_2 &= -\tan^{-1}\frac{2}{3} = -0.588(3dp) \end{aligned}$$

This means that
$z_2 = 3 - j2 = \sqrt{13}(\cos(-0.588) + j\sin(-0.588)) = \sqrt{13}(\cos 0.588 - j\sin 0.588)$

Multiplication in polar form

Let $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$ then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

which by using trigonometric identities gives

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)]$$

Hence

$$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2|$$

and

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

When using these results care must be taken to ensure that $-\pi < \arg(z_1 z_2) \leq \pi$.

Effect of multiplying by j

Polar form helps to make explicit the effect of multiplying by j.

Consider the complex number $z = r(\cos\theta + j\sin\theta)$.

By the Argand diagram we see that j can be written as $j = 1(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2})$ it then follows that

$$jz = r[\cos(\theta + \frac{\pi}{2}) + j\sin(\theta + \frac{\pi}{2})]$$

Thus the effect of multiplying a complex number by j is

- To leave the modulus unaltered.
- To increase the argument by $\frac{\pi}{2}$.

This property is of importance in the application of complex numbers to the theory of alternating current.

Example

Question

For $z_1 = 2 + j3$ and $z_2 = 3 - j2$ determine $|z_1 z_2|$ and $\arg(z_1 z_2)$

Solution

For z_1 we have

$$
\begin{aligned}
|z_1| &= \sqrt{2^2 + 3^2} = \sqrt{13} \\
\arg z_1 &= \tan^{-1}\frac{3}{2} = 0.982(3dp)
\end{aligned}
$$

and for z_2 we have

$$
\begin{aligned}
|z_2| &= \sqrt{3^2 + (-2)^2} = \sqrt{13} \\
\arg z_2 &= -\tan^{-1}\frac{2}{3} = -0.588(3dp)
\end{aligned}
$$

This means that $|z_1 z_2| = 13$ and $\arg(z_1 z_2) = 0.394$

Example

Question
For $z_1 = 2 + j3$ and $z_2 = 3 - j2$ determine $|z_1 z_2|$ and arg $(z_1 z_2)$
Solution
For z_1 we have

$$\begin{aligned} |z_1| &= \sqrt{2^2 + 3^2} = \sqrt{13} \\ \arg z_1 &= \tan^{-1}\frac{3}{2} = 0.982(3dp) \end{aligned}$$

and for z_2 we have

$$\begin{aligned} |z_2| &= \sqrt{3^2 + (-2)^2} = \sqrt{13} \\ \arg z_2 &= -\tan^{-1}\frac{2}{3} = -0.588(3dp) \end{aligned}$$

This means that $|z_1 z_2| = 13$ and arg $(z_1 z_2) = 0.394$

Division in Polar Form

First consider $\frac{1}{z}$ where $z = \cos\theta + j\sin\theta$

$$\begin{aligned}
\frac{1}{\cos\theta + j\sin\theta} &= \frac{1}{\cos\theta + j\sin\theta}\,\frac{\cos\theta - j\sin\theta}{\cos\theta - j\sin\theta} \\
&= \frac{\cos\theta - j\sin\theta}{\cos^2\theta + \sin^2\theta} \\
&= \cos\theta - j\sin\theta \qquad \text{since } \cos^2\theta + \sin^2\theta = 1
\end{aligned}$$

We can use this to work out $\frac{z_1}{z_2}$ where $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)} \\
&= \frac{r_1}{r_2}(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 - j\sin\theta_2) \\
&= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)] \\
&= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)]
\end{aligned}$$

where a trigonometric identity was used in the final step.

Division in Polar Form

We showed that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)]$$

Hence

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$$

Example

Question

For $z_1 = 2 - j4$ and $z_2 = 3 + j2$ determine $\left|\frac{z_1}{z_2}\right|$ and arg $\left(\frac{z_1}{z_2}\right)$

Solution

For z_1 we have

$$\begin{aligned} |z_1| &= \sqrt{2^2 + (-4)^2} = \sqrt{20} \\ \text{arg } z_1 &= -\tan^{-1}\frac{4}{2} = -1.107(3dp) \end{aligned}$$

and for z_2 we have

$$\begin{aligned} |z_2| &= \sqrt{3^2 + (2)^2} = \sqrt{13} \\ \text{arg } z_2 &= \tan^{-1}\frac{2}{3} = 0.588(3dp) \end{aligned}$$

This means that $\left|\frac{z_1}{z_2}\right| = \sqrt{20/13}$ and arg $\left(\frac{z_1}{z_2}\right) = -1.695$ rad.

Example

Question

For $z_1 = 2 - j4$ and $z_2 = 3 + j2$ determine $\left|\frac{z_1}{z_2}\right|$ and $\arg\left(\frac{z_1}{z_2}\right)$

Solution

For z_1 we have

$$\begin{aligned} |z_1| &= \sqrt{2^2 + (-4)^2} = \sqrt{20} \\ \arg z_1 &= -\tan^{-1}\frac{4}{2} = -1.107(3dp) \end{aligned}$$

and for z_2 we have

$$\begin{aligned} |z_2| &= \sqrt{3^2 + (2)^2} = \sqrt{13} \\ \arg z_2 &= \tan^{-1}\frac{2}{3} = 0.588(3dp) \end{aligned}$$

This means that $\left|\frac{z_1}{z_2}\right| = \sqrt{20/13}$ and $\arg\left(\frac{z_1}{z_2}\right) = -1.695$ rad.

Outline

Euler's Formula

Euler's formula expresses the very useful result that

$$e^{jy} = \cos y + j \sin y$$

This result is very useful and it allows to introduce the **exponential form** of the complex number

$$z = re^{j\theta}$$

which follows since $z = x + jy = r(\cos\theta + j\sin\theta) = re^{j\theta}$ where r is the modulus of z and θ is the argument of z

Proof of Euler's Formula

There number of alternative techniques for proving Euler's formula, we use MacLaurin series:

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots$$
$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots$$
$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Substitute $z = jy$ in the last series

$$\begin{aligned} e^z &= 1 + jy + \frac{(jy)^2}{2!} + \frac{(jy)^3}{3!} + \cdots \\ &= 1 + jy - \frac{y^2}{2!} - \frac{jy^3}{3!} + \frac{y^4}{4!} + \frac{jy^5}{5!} - \cdots \\ &= \cos y + j \sin y \end{aligned}$$

Relating Circular and Hyperbolic Functions

Recall the definitions of the hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

From Euler's formula we have

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad e^{-j\theta} = \cos\theta - j\sin\theta$$

so that

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Comparing we can write

$$\cosh jx = \frac{e^{jx} + e^{-jx}}{2} = \cos x \qquad \sinh jx = \frac{e^{jx} - e^{-jx}}{2} = j\sin x$$

so that

$$\tanh jx = \frac{\sinh jx}{\cosh jx} = j\frac{\sin x}{\cos x} = j\tan x$$

Relating Circular and Hyperbolic Functions

Alternatively choosing $\theta = jx$ in $\cos\theta$ and $\sin\theta$ on the previous slide gives

$$\cos jx = \frac{e^{j^2x} + e^{-j^2x}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x$$

$$\sin jx = \frac{e^{j^2x} - e^{-j^2x}}{2j} = \frac{e^{-x} - e^{x}}{2j} = j\sinh x$$

so that

$$\tan jx = \frac{\sin jx}{\cos jx} = j\frac{\sinh x}{\cosh x} = j\tanh x$$

Example

Question
Find the value of $\sin[\frac{\pi}{4}(1+j)]$
Solution
We can initially use the identity $\sin(A+B) = \sin A\cos B + \cos A \sin B$ to give

$$\sin[\frac{\pi}{4}(1+j)] = \sin\frac{\pi}{4}\cos j\frac{\pi}{4} + \cos\frac{\pi}{4}\sin j\frac{\pi}{4}$$

We can directly evaluate $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \sqrt{\frac{1}{2}}$ and make use of $\cos jx = \cosh x$ and $\sin jx = j\sinh x$ to further simplify the result.

$$\begin{aligned}\sin[\frac{\pi}{4}(1+j)] &= \sqrt{\frac{1}{2}}\cos j\frac{\pi}{4} + \sqrt{\frac{1}{2}}\sin j\frac{\pi}{4} \\ &= \sqrt{\frac{1}{2}}\cosh\frac{\pi}{4} + j\sqrt{\frac{1}{2}}\sinh\frac{\pi}{4} \\ &= 0.937 + j0.614\end{aligned}$$

Example

Question

Find the value of $\sin[\frac{\pi}{4}(1+j)]$

Solution

We can initially use the identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$ to give

$$\sin[\frac{\pi}{4}(1+j)] = \sin\frac{\pi}{4}\cos j\frac{\pi}{4} + \cos\frac{\pi}{4}\sin j\frac{\pi}{4}$$

We can directly evaluate $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \sqrt{\frac{1}{2}}$ and make use of $\cos jx = \cosh x$ and $\sin jx = j\sinh x$ to further simplify the result.

$$\begin{aligned}
\sin[\frac{\pi}{4}(1+j)] &= \sqrt{\frac{1}{2}}\cos j\frac{\pi}{4} + \sqrt{\frac{1}{2}}\sin j\frac{\pi}{4} \\
&= \sqrt{\frac{1}{2}}\cosh\frac{\pi}{4} + j\sqrt{\frac{1}{2}}\sinh\frac{\pi}{4} \\
&= 0.937 + j0.614
\end{aligned}$$

Example

Question
Find the value of $\cos^{-1}\frac{3j}{4}$
Solution
We first write $\cos^{-1}\frac{3j}{4} = x + jy$ and so $\frac{3j}{4} = \cos(x + jy)$. Expanding

$$\cos(x + jy) = \cos x \cos jy - \sin x \sin jy = \cos x \cosh y - j \sin \sinh y$$

Equating real and imaginary parts

$$0 = \cos x \cosh y \qquad -\frac{3}{4} = \sin x \sinh y$$

To find the solution we need to solve these two non-linear equations for x and y. It is easier to start with the one equal to zero.

Either $\cosh y = 0$ or $\cos x = 0$. But $\cosh y \neq 0$ for real y and so $\cos x = 0$. Thus $x = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$.

Now $\sin x = \sin(\frac{\pi}{2} + k\pi) = (-1)^k$ and so $\sinh y = -\frac{3(-1)^k}{4}$. If k is even $y \approx -0.693$ and if k is odd, $y \approx 0.693$. So $y \approx -0.693(-1)^k$.

Finally

$$\cos^{-1}\frac{3j}{4} = x + jy = \frac{\pi}{2} + k\pi - j0.693(-1)^k, \qquad k \in \mathbb{Z}$$

Example

Question

Find the value of $\cos^{-1}\frac{3j}{4}$

Solution

We first write $\cos^{-1}\frac{3j}{4} = x + jy$ and so $\frac{3j}{4} = \cos(x + jy)$. Expanding

$$\cos(x + jy) = \cos x \cos jy - \sin x \sin jy = \cos x \cosh y - j \sin \sinh y$$

Equating real and imaginary parts

$$0 = \cos x \cosh y \qquad -\frac{3}{4} = \sin x \sinh y$$

To find the solution we need to solve these two non-linear equations for x and y. It is easier to start with the one equal to zero.

Either $\cosh y = 0$ or $\cos x = 0$. But $\cosh y \neq 0$ for real y and so $\cos x = 0$. Thus $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

Now $\sin x = \sin(\frac{\pi}{2} + k\pi) = (-1)^k$ and so $\sinh y = -\frac{3(-1)^k}{4}$. If k is even $y \approx -0.693$ and if k is odd, $y \approx 0.693$. So $y \approx -0.693(-1)^k$.

Finally

$$\cos^{-1}\frac{3j}{4} = x + jy = \frac{\pi}{2} + k\pi - j0.693(-1)^k, \qquad k \in \mathbb{Z}$$

Logarithms

If $w = \ln z$ then $z = e^w$. Writing $z = x + jy$ and $w = u + jv$ we have

$$x + jy = e^{u+jv} = e^u e^{jv} = e^u(\cos v + j \sin v)$$

by Euler's formula. We need to find expressions for u and v.

Equating real and imaginary parts

$$x = e^u \cos v \qquad y = e^u \sin v$$

Squaring and adding both these equations

$$|z|^2 = x^2 + y^2 = e^{2u}(\cos^2 v + \sin^2 v) = e^{2u}$$

so that

$$u = \frac{1}{2} \ln(x^2 + y^2) = \ln |z|$$

Logarithms

To obtain v, we first divide the equation for the imaginary part by the equation for the real part to obtain

$$\tan v = \frac{y}{x}$$

Thus

$$v = \text{arg}\, z$$

and finally

$$\ln z = \ln |z| + j\text{arg}\, z$$

Example

Question
Evaluate $\ln(3+j4)$ in the form $x+jy$
Solution

$$\begin{aligned} |3+j4| &= \sqrt{3^2+4^2} = \sqrt{25} = 5 \\ \text{arg}\,(3+j4) &= \tan^{-1}\frac{4}{3} = 0.927 \end{aligned}$$

so that

$$\ln(3+j4) = \ln 5 + j0.927 = 1.609 + j0.927$$

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Evaluate $\ln(3 + j4)$ in the form $x + jy$

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Powers of Sine and Cosine

We can use Euler's formula to aid us with deriving trigonometric identities

If $z = \cos\theta + j\sin\theta$

$$z^n = \cos n\theta + j\sin n\theta \qquad z^{-n} = \cos n\theta - j\sin n\theta$$

so that

$$z^n + z^{-n} = 2\cos n\theta \qquad z^n - z^{-n} = 2j\sin n\theta$$

These results are very useful as this next example shows

Example

Question

Express $\cos^5\theta$ in terms of multiple angles

Solution

We can use $z^n + z^{-n} = 2\cos n\theta$ with $n = 1$ to obtain

$$(2\cos\theta)^5 = \left(z + \frac{1}{z}\right)^5 = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$$

so that

$$32\cos^5\theta = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

and finally using $z^n + z^{-n} = 2\cos n\theta$ again but now with $n = 5, 3, 1$ we have

$$\cos^5\theta = \frac{1}{32}(2\cos 5\theta + 10\cos 3\theta + 20\cos\theta)$$

Example

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Express $\cos^5\theta$ in terms of multiple angles

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We can use $z^n + z^{-n} = 2\cos n\theta$ with $n = 1$ to obtain

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Outline

De Moivre's Theorem

We have all the tools at our disposal to write the following simple result for $z = re^{j\theta}$:

$$z^n = r^n (e^{j\theta})^n = r^n e^{j(n\theta)}$$

so that

$$z^n = r^n (\cos n\theta + j \sin n\theta)$$

This is known as **de Moivre's theorem**.

Example

Question
Express $1+j$ in the form $r(\cos\theta + j\sin\theta)$ and hence find $(1+j)^4$
Solution

$$
\begin{aligned}
|1+j| &= \sqrt{1^2+1^2} = \sqrt{2} \\
\text{arg}\,(1+j) &= \tan^{-1}\frac{1}{1} = \frac{\pi}{4}
\end{aligned}
$$

so that $1+j = \sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$ which means that

$$
(1+j)^4 = (\sqrt{2})^4\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)^4
$$

Now using de Moivre's theorem we have

$$
\begin{aligned}
(1+j)^4 &= (\sqrt{2})^4\left(\cos 4\frac{\pi}{4} + j\sin 4\frac{\pi}{4}\right) \\
&= 4(-1+j0) \\
&= -4
\end{aligned}
$$

Example

Question

Express $1+j$ in the form $r(\cos\theta + j\sin\theta)$ and hence find $(1+j)^4$

Solution

$$\begin{aligned} |1+j| &= \sqrt{1^2+1^2} = \sqrt{2} \\ \text{arg}\,(1+j) &= \tan^{-1}\frac{1}{1} = \frac{\pi}{4} \end{aligned}$$

so that $1+j = \sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$ which means that

$$(1+j)^4 = (\sqrt{2})^4\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)^4$$

Now using de Moivre's theorem we have

$$\begin{aligned} (1+j)^4 &= (\sqrt{2})^4\left(\cos 4\frac{\pi}{4} + j\sin 4\frac{\pi}{4}\right) \\ &= 4(-1+j0) \\ &= -4 \end{aligned}$$