Engineering Analysis 2

Lecture Notes Written by Dr. P.D. Ledger

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Contents

Chapter 1

Vector Algebra

Vectors are used when ever we wish to describe an engineering problem in two or three–dimensions. They are closely related with coordinate geometry and this is where our discussions begin. We shall learn how to work with vectors, how they can multiplied together and used to define the equations of lines and planes. Although we shall restrict consideration to vectors in two or three dimensions, they are also important in higher dimensions, as previously seen in the context of linear algebra.

1.1 Vectors and Scalars

A **vector** is a quantity which has both magnitude as well as direction. In these notes, we shall use the notation $\boldsymbol{u}$ to denote that the quantity is a vector. Note that in other textbooks, authors may also use the symbols $\vec{u}$ or $\underline{u}$ to distinguish vector quantities. A **scalar** is a quantity which has magnitude only, we shall use the plain notation ϕ to denote that the quantity is a scalar.

A vector may be drawn as shown in Figure 1.1.

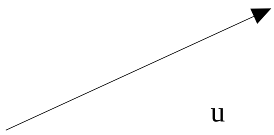


Figure 1.1: Illustration of a vector

1.2 Addition of Vectors

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ may be added to give a new vector $\boldsymbol{a}+\boldsymbol{b}$ as illustrated in Figure 1.2. Note that the vector $-\boldsymbol{a}$ is the vector with equal magnitude to that of $\boldsymbol{a}$, but with opposite direction. Adding $\boldsymbol{a}$ and $-\boldsymbol{a}$ gives the $\mathbf{0}$ vector which has zero magnitude and so has no direction. Nevertheless it is sensible to regard $\mathbf{0}$ as a vector.

1.3 Components of a Vector

Vectors are often written using the **Cartesian coordinate system**, as illustrated in Figure 1.3. This coordinate systems follows the convention of being **right handed**. Which means that in the sense that a rotation of right handed screw from Ox to Oy advances it along Oz. A rotation from

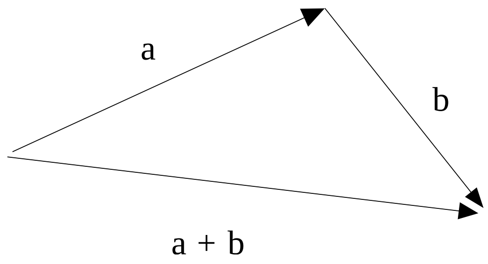


Figure 1.2: Addition of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

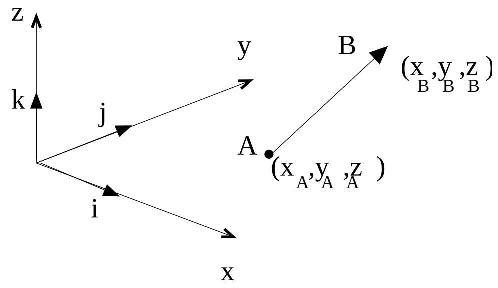


Figure 1.3: Components of the vector $\boldsymbol{a} = \vec{AB}$ expressed in Cartesian coordinates

Oy to Oz advances it along Ox and a rotation from Oz to Ox advances it along Oy. This is a widely adopted convention and we shall use it throughout this chapter. In this figure we also see how a vector $\boldsymbol{a}$ extends from the point A with x, y, z coordinates (x_A, y_A, z_A) to the point B with coordinates (x_B, y_B, z_B). It is common to write $\boldsymbol{a} = \vec{AB}$. The **components** of the vector are defined to be $a_1 = x_B - x_A$, $a_2 = y_B - y_A$ and $a_3 = z_B - z_A$ and the vector can then be written in the form

$$\boldsymbol{a} = \vec{AB} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix} \tag{1.1}$$

Two vectors are said to be **equal** if the components of both vectors are the same. We can multiply a vector by a scalar to get another vector. If $\boldsymbol{a} = \lambda \boldsymbol{b}$ where λ is some scalar quantity, then if $\lambda > 0$ the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be **parallel** and if $\lambda < 0$ the vectors are said to be **anti–parallel**.

The **magnitude** or length of a vector is given $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. A vector with unit magnitude is called the **unit vector**. A unit vector $\hat{\boldsymbol{a}}$ is obtain from a vector $\boldsymbol{a}$ using the formula

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|} \tag{1.2}$$

By defining the unit vectors

$$\boldsymbol{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.3}$$

which point along the x, y and z axis, respectively, the vector $\boldsymbol{a}$ may be written as

$$\boldsymbol{a} = a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k} \tag{1.4}$$

and we shall adopt this convention throughout this chapter.

Example
Determine the vector $\boldsymbol{a} = \vec{OA}$ where O is the origin and A is the point with coordinates $(-1,-2,-3)$
Solution
$\boldsymbol{a} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = -1\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$

This notation allows a neat way of expressing the addition of two vectors. For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with components a_1, a_2, a_3 and b_1, b_2, b_3 the addition of the two vectors can be written as

$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j} + (a_3 + b_3)\boldsymbol{k} \tag{1.5}$$

Example (Exercise Class)
Determine the vector $\boldsymbol{a}+\boldsymbol{b}$ where $\boldsymbol{a} = \vec{OA}$, $\boldsymbol{b} = \vec{OB}$ with O the origin, A the point $(-1,-2,-3)$ and B the point $(2,1,3)$. Subsequently determine $|\boldsymbol{a}+\boldsymbol{b}|$
Solution
$\boldsymbol{a} = -1\boldsymbol{i} - 2\boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$. Thus

$$\boldsymbol{a} + \boldsymbol{b} = (-1+2)\boldsymbol{i} + (-2+1)\boldsymbol{j} + (-3+3)\boldsymbol{k} = \boldsymbol{i} - \boldsymbol{j}$$

Also $|\boldsymbol{a}+\boldsymbol{b}| = \sqrt{(1)^2 + (-1)^2 + 0^2} = \sqrt{2}$

1.4 Direction Cosines

By considering a vector $\boldsymbol{r} = \vec{OP}$ which points from the origin O to some point P with coordinates (x_P, y_P, z_P) it is possible to compute its **direction cosines**. If the vector $\boldsymbol{r}$ has modulus $r = |\boldsymbol{r}|$ then its direction cosines are defined as

$$l = \cos\alpha = \frac{x_P}{r} \qquad m = \cos\beta = \frac{y_P}{r} \qquad n = \cos\gamma = \frac{z_P}{r} \tag{1.6}$$

where α, β, γ are the angles that the vector makes with the x, y and z axes, respectively (see Figure 1.4). The direction cosines satisfy the property

$$l^2 + m^2 + n^2 = \frac{x_P^2}{r^2} + \frac{y_P^2}{r^2} + \frac{z_P^2}{r^2} = \frac{x_P^2 + y_P^2 + z_P^2}{r^2} = 1 \tag{1.7}$$

Direction cosines are frequently used in surveying.

Example
If P has coordinates $(2,-1,3)$, find the direction cosines of $\vec{OP}$
Solution
$r = |\vec{OP}| = \sqrt{(2)^2 + (-1)^2 + (3)^2} = \sqrt{14}$ and thus the direction cosines are

$$l = \frac{2}{\sqrt{14}} \qquad m = -\frac{1}{\sqrt{14}} \qquad n = \frac{3}{\sqrt{14}}$$

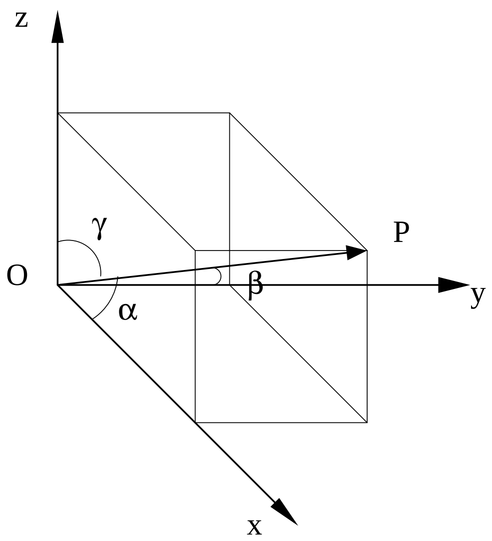


Figure 1.4: Illustration of the angles α, β, γ relating to the direction cosines

1.5 Dot Product

The **dot product** or scalar product of two vectors is a scalar quantity, it is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta \tag{1.8}$$

where θ is the angle between the two vectors. The dot product has the following properties

- The dot product is commutative $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$;
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** (orthogonal) $\boldsymbol{a} \cdot \boldsymbol{b} = 0$;
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular or $\boldsymbol{a}$ or $\boldsymbol{b}$ or both are zero;
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$;
- As shown in Figure 1.5, $\boldsymbol{a} \cdot \boldsymbol{b}$ is the magnitude of $\boldsymbol{a}$ multiplied by the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.
- The dot product is distributive over addition $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$.

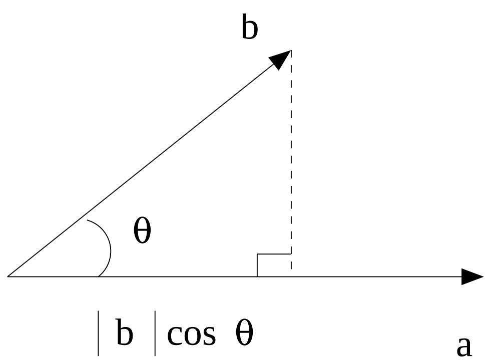


Figure 1.5: Physical realisation of the dot product $\boldsymbol{a} \cdot \boldsymbol{b}$

We may write a formula for the dot product in terms of the Cartesian component of a vector as follows: First note that $\boldsymbol{i} \cdot \boldsymbol{i} = 1$, $\boldsymbol{j} \cdot \boldsymbol{j} = 1$, $\boldsymbol{k} \cdot \boldsymbol{k} = 1$ and that $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0$, then

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \cdot (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) \\ &= a_1b_1\boldsymbol{i} \cdot \boldsymbol{i} + a_2b_2\boldsymbol{j} \cdot \boldsymbol{j} + a_3b_3\boldsymbol{k} \cdot \boldsymbol{k} \\ &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned} \tag{1.9}$$

Example
Given the vectors $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j}$ determine a) $\boldsymbol{a} \cdot \boldsymbol{b}$ and b) $\boldsymbol{b} \cdot (\boldsymbol{a} + 2\boldsymbol{c})$
Solution
a) Direct substitution gives

$$\boldsymbol{a} \cdot \boldsymbol{b} = 1(1) + 1(-1) + 2(2)\boldsymbol{k} = 4$$

b) First compute $\boldsymbol{a} + 2\boldsymbol{c}$

$$\boldsymbol{a} + 2\boldsymbol{c} = (1 + 2(1))\boldsymbol{i} + (-1 + 2(1))\boldsymbol{j} + (2 + 2(0))\boldsymbol{k} = 3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$$

Then

$$\boldsymbol{b} \cdot (\boldsymbol{a} + 2\boldsymbol{c}) = 1(3) + 1(1) + 2(2) = 8$$

Example (Exercise Class)
Find the angle between the vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + 4\boldsymbol{k}$
Solution
By definition $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$. Here

$$\boldsymbol{a} \cdot \boldsymbol{b} = 1(2) + 2(0) + 3(4) = 14$$

Also

$$|\boldsymbol{a}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14} \qquad |\boldsymbol{b}| = \sqrt{(2)^2 + (0)^2 + (4)^2} = \sqrt{20}$$

Giving the final result

$$14 = \sqrt{14}\sqrt{20}\cos\theta \qquad \theta \approx 0.580rad$$

1.6 Cross Product

The **cross product** or vector product between two vectors is a vector quantity $\boldsymbol{a} \times \boldsymbol{b}$. It has magnitude and direction. Its magnitude is $|\boldsymbol{a}||\boldsymbol{b}| \sin\theta$ where θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ expressed in the same manner as the Cartesian coordinate system. Thus $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \boldsymbol{v}$ where $\boldsymbol{v}$ is a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

The cross product has the following properties

- The cross product is not commutative $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$;
- If the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$;
- The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is the area of parallelogram made by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$;

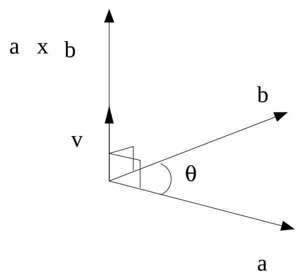


Figure 1.6: Physical realisation of the cross product $\boldsymbol{a} \times \boldsymbol{b}$

The cross product may also be written in terms of components. Consider first $\boldsymbol{i} \times \boldsymbol{j}$, since these two vectors have magnitude 1 and are perpendicular $\sin\theta = 1$ and therefore the magnitude of $\boldsymbol{i} \times \boldsymbol{j}$ is 1. The direction of $\boldsymbol{i} \times \boldsymbol{j}$ perpendicular to both $\boldsymbol{i}$ and $\boldsymbol{j}$ and so $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$. It follows that $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{0}$, $\boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{0}$, $\boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$ and $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$.

Thus

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}) \times (b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}) \\
&= a_1b_2\boldsymbol{i} \times \boldsymbol{j} + a_1b_3\boldsymbol{i} \times \boldsymbol{k} + a_2b_1\boldsymbol{j} \times \boldsymbol{i} + a_2b_3\boldsymbol{j} \times \boldsymbol{k} + \\
&\qquad a_3b_1\boldsymbol{k} \times \boldsymbol{i} + a_3b_2\boldsymbol{k} \times \boldsymbol{j} \\
&= (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}, \qquad (1.10)
\end{aligned}$$

1.6.1 Cross product using determinants

An easy way to remember the cross product is using a **determinate**. The determinate is a square array of numbers expressing sums of products of numbers. A 2×2 determinate is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad (1.11)$$

for example

$$\begin{vmatrix} 3 & -2 \\ -2 & -1 \end{vmatrix} = 3(-1) - (-2)(-2) = -7$$

A 3×3 determinate is given by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \qquad (1.12)$$

for example

$$\begin{aligned}
\begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 1 \end{vmatrix} &= 1\begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} \\
&= 1(0(1) - (-1)(1)) + ((1)(1) - (-1)(-2)) + 2((1)(1) - 0(-2)) \\
&= 1 - 1 + 2 \\
&= 2
\end{aligned}$$

The vector product of $\boldsymbol{a} \times \boldsymbol{b}$ with $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is equivalent to

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \boldsymbol{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \qquad (1.13)$$

1.7 Scalar Triple Product

The scalar triple product is defined as $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ and is a scalar quantity. The scalar triple product has the following properties

- The dot and the cross may be interchanged $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$;
- The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ may be permuted cyclically $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})$;

Using the previous results, we have that

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \tag{1.14}$$

1.7.1 Scalar triple product using determinants

An easy way to remember the scalar triple product of $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ is using the determinate

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{1.15}$$

1.8 Vector Triple Product

The vector triple product $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ is a vector quantity. Note that the brackets are important here as $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) \neq (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$. The vector triple product has the following properties

- $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$;
- $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = -\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) = -(\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a} + (\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b}$.

Example
For the vectors $\boldsymbol{a} = 1\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = 1\boldsymbol{i} + 1\boldsymbol{j} + 1\boldsymbol{k}$ and $\boldsymbol{c} = 1\boldsymbol{i} + 0\boldsymbol{j} + 3\boldsymbol{k}$. Determine $\boldsymbol{a}\times\boldsymbol{b}$, $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ and $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$
Solution
By direct substitution

$$\boldsymbol{a}\times\boldsymbol{b} = (2(1) - 3(1))\boldsymbol{i} + (3(1) - 1(1))\boldsymbol{j} + (1(1) - 2(1))\boldsymbol{k} = -\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$

(Alternatively the determinate method might be used)
To obtain $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$, first compute $\boldsymbol{b}\times\boldsymbol{c}$ to give

$$\boldsymbol{b}\times\boldsymbol{c} = (1(3) - 1(0))\boldsymbol{i} + (1(1) - 1(3))\boldsymbol{j} + (1(0) - 1(1))\boldsymbol{k} = 3\boldsymbol{i} - 2\boldsymbol{j} - 1\boldsymbol{k}$$

Then

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = 1(3) + 2(-2) + 3(-1) = -4$$

(Alternatively the determinate method might be used)
For $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$, we use the result for $\boldsymbol{b}\times\boldsymbol{c}$ and find

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (2(-1) - 3(-2))\boldsymbol{i} + (3(3) - 1(-1))\boldsymbol{j} + (1(-2) - 2(3))\boldsymbol{k} = 4\boldsymbol{i} + 10\boldsymbol{j} - 8\boldsymbol{k}$$

1.9 Equation of a Line

Given a point A on a line and direction vector $\boldsymbol{d}$ for the line, it is possible to construct the general equation of a line. The direction vector can be any vector that is parallel to the line.

Let $\boldsymbol{d} = l\boldsymbol{i} + m\boldsymbol{j} + n\boldsymbol{k}$ be a known direction vector and $A = (x_a, y_A, z_A)$ be a given point and $W = (x, y, z)$ a general point on the line. Set $\boldsymbol{a} = \vec{OA}$ and $\boldsymbol{w} = \vec{OW}$, as illustrated in Figure 1.7.

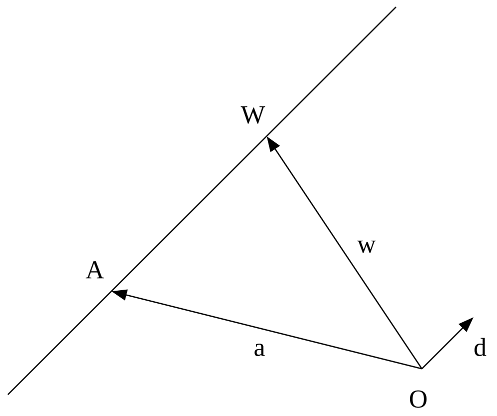


Figure 1.7: Illustration of line, its direction vector, and two point lying on the line

The vector $\vec{AW} = \boldsymbol{w} - \boldsymbol{a}$ is parallel to the line and so must be a scalar multiple of $\boldsymbol{d}$. So

$$\boldsymbol{w} - \boldsymbol{a} = t\boldsymbol{d} \tag{1.16}$$

for some t. Hence $\boldsymbol{w} = \boldsymbol{a} + t\boldsymbol{d}$ which is the **vector from of the equation of the line** where t can be any scalar. Equating components

$$x = x_A + tl \qquad y = y_A + tm \qquad z = z_A + tn \tag{1.17}$$

Then, provided that l, m, n are non-zero, the **Cartesian form of the equation of the line** is

$$\frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n} \qquad (= t) \tag{1.18}$$

Conversely if the equation of the line is known as

$$\frac{x - x_A}{l} = \frac{y - y_A}{m} = \frac{z - z_A}{n} \qquad (= t) \tag{1.19}$$

then (x_A, y_A, z_A) is a point on the line and $\boldsymbol{d} = l\boldsymbol{i} + m\boldsymbol{j} + n\boldsymbol{k}$ is a direction vector for the line.

If the direction vector is not known and instead two points A and B which lie on the line are prescribed, a direction vector is given by $\boldsymbol{d} = \vec{AB}$ as $\vec{AB}$ is parallel to the line.

Example
Determine the equation of the line containing the points A with coordinates $(-1,-2,3)$ and B with coordinates $(1,1,2)$.
Solution
A direction vector for the line is $\boldsymbol{d} = \vec{AB} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$. Also $\boldsymbol{a} = \vec{OA} = -\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$
The equation of the line in vector form is

$$\begin{aligned}
\boldsymbol{w} &= \boldsymbol{a} + t\boldsymbol{d} = (-\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) + t(2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}) \\
\boldsymbol{w} &= (-1+2t)\boldsymbol{i} + (-2+3t)\boldsymbol{j} + (3-t)\boldsymbol{k}
\end{aligned}$$

The Cartesian form of the equation of the line is

$$\frac{x+1}{2} = \frac{y+2}{3} = \frac{z-3}{-1}$$

1.10 Equation of a Plane

A **normal vector** $\boldsymbol{n}$ is a vector which is perpendicular to the plane. Let $\boldsymbol{n} = n_1\boldsymbol{i} + n_2\boldsymbol{j} + n_3\boldsymbol{k}$ be a normal vector and $A = (x_A, y_A, z_A)$ be a point on a plane and $W = (x, y, z)$ be a general point on the plane as shown on Figure 1.8

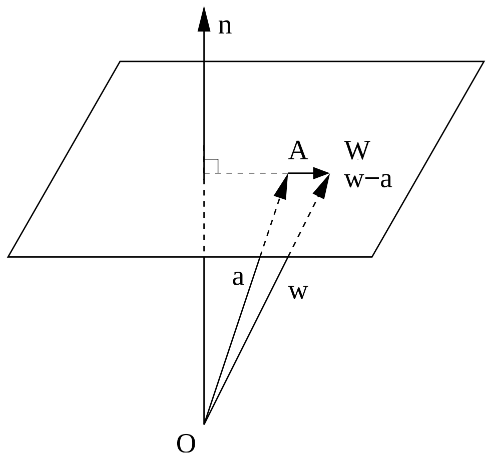


Figure 1.8: Illustration of a plane, its normal vector, and two point lying on it

Let $\boldsymbol{a} = \vec{OA} = x_A\boldsymbol{i} + y_A\boldsymbol{j} + z_A\boldsymbol{k}$ and $\boldsymbol{w} = \vec{OW} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Then $\vec{AW} = \boldsymbol{w} - \boldsymbol{a}$ is parallel to the line AW which lies in the plane and so is perpendicular to $\boldsymbol{n}$. Hence

$$\boldsymbol{n} \cdot (\boldsymbol{w} - \boldsymbol{a}) = 0 \qquad (1.20)$$

The **vector form of the equation of a plane** is

$$\boldsymbol{n} \cdot \boldsymbol{w} = \boldsymbol{n} \cdot \boldsymbol{a} = p \qquad (1.21)$$

where p represents the perpendicular distance from, the origin to the plane. The **Cartesian form of the equation of a plane** is

$$n_1x + n_2y + n_3z = p \qquad (1.22)$$

Example
Find the equation of the plane containing the points $A = (1,1,1)$, $B = (0,1,2)$ and $C = (-1,1,-1)$.
Solution
First we construct the vectors $\boldsymbol{a} = \vec{OA} = \boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$, $\boldsymbol{b} = \vec{OB} = \boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{c} = \vec{OC} = -\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$. The vectors $\boldsymbol{a}-\boldsymbol{b} = \boldsymbol{i}-\boldsymbol{k}$ and $\boldsymbol{a}-\boldsymbol{c} = 2\boldsymbol{i}+2\boldsymbol{k}$ lie in the plane. The normal vector can be constructed as

$$\boldsymbol{n} = (\boldsymbol{a}-\boldsymbol{b}) \times (\boldsymbol{a}-\boldsymbol{c}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -1 \\ 2 & 0 & 2 \end{vmatrix} = -4\boldsymbol{j}$$

The vector form of the equation of the plane is

$$\boldsymbol{w} \cdot (-4\boldsymbol{j}) = (\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) \cdot (-4\boldsymbol{j})$$

which in Cartesian form is

$$0x - 4y + 0z = 1(0) + 1(-4) + 1(0)$$

or simply $y = 1$

1.11 Summary and Further Reading

The key topics from this chapter and references to further reading in James (fourth edition) and Croft and Davison (second edition) are

- Addition and components of a vector. Croft and Davison [pg 627-654]. James [pg 230-231, 233-247]
- Direction cosines. Croft and Davison [pg. 654-656]. James [pg 231-232]
- Scalar product. Croft and Davison [pg 659-670]. James [pg 251-258].
- Cross product. Croft and Davison [pg. 671-681]. James [258-268].
- Triple products. James [269-275].
- Equation of a line. Croft and Davison [pg 681-686]. James [pg. 276-283].
- Equation of a plane. Croft and Davison [pg 687-691]. James [pg. 283-287].

Chapter 2

Complex Numbers

The numbers we have encountered so far within these notes have been real numbers. In order to solve certain types of mathematical problems it is necessary to introduce further numbers. These numbers are called complex numbers. An important application of complex numbers is in the analysis of alternating current circuits. In this chapter we shall introduce some of the proprieties of complex numbers and explain how to work with them.

2.1 The Number j

We know that when we square a positive or negative number the result is always positive, for example $3^2 = 9$ and $(-3)^2 = 9$. Let us now suppose that we would like to determine $\sqrt{-9}$, unfortunately the mathematics we have learnt up until now will not allow us to perform this operation, as the square root only makes sense for positive real numbers. For certain applications it is useful to overcome this limitation. To do this we introduce a new number, to which we will give the symbol j, which has the property that

$$j^2 = -1 \qquad \text{so that } j = \sqrt{-1} \tag{2.1}$$

As no real number when squared equals -1, the number j cannot be real. Instead we call it an **imaginary** number. Although the concept of an imaginary number may seem strange at first, in turns out to be very useful in engineering applications. Mathematicians and physicists often prefer the symbol i instead of the symbol j for $\sqrt{-1}$.

Using this notation we now in a position to write down the square root of any negative number, for example $\sqrt{-9} = j3$, the result is not a real number, but instead an imaginary number.

2.2 The Complex Number $a + jb$

In the first chapter we derived the formula for obtaining general roots to the quadratic equation $ax^2 + bx + c = 0$, namely

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.2}$$

As soon as $4ac > b^2$, the term $b^2 - 4ac$ will become negative and thus the above expression will involve the square root of a negative number. In this case we can use j to help us evaluate the square root. For example the quadratic equation $x^2 - 6x + 10 = 0$ has the roots $x = 3 \pm j$. Thus we can write down the solutions of the equation as $3 + j$ and $3 - j$, these two numbers are called **complex numbers**. Each number consists of two parts: a **real part** and an **imaginary part**. The set of all complex numbers is given the symbol $\mathbb{C}$.

In general we give complex numbers the symbol z and write them as

$$z = a + jb \tag{2.3}$$

where $a = \mathrm{Re}(z)$ is the real part of the complex number and $b = \mathrm{Im}(z)$ is its imaginary part. When the complex number is written in this way it is called its **Cartesian form**.

Example
Determine the roots of the quadratic equation

$$x^2 - 3x + 4 = 0 \tag{2.4}$$

Solution
Applying equation (2.2) we have

$$x = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm \sqrt{-7}}{2} = \frac{3 \pm j\sqrt{7}}{2} \tag{2.5}$$

here we used $j = \sqrt{-1}$ to rewrite $\sqrt{-7}$ as $j\sqrt{7}$.

2.3 Manipulation of Complex Numbers

To add or subtract two complex numbers we simply perform the operations on their respective real and imaginary parts. For example, if $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ then

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{2.6}$$

and

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{2.7}$$

For the multiplication of complex numbers we make use of the fact that $j^2 = -1$

$$\begin{aligned} z_1 z_2 = (x_1 + jy_1) \times (x_2 + jy_2) &= x_1x_2 + jx_1y_2 + jx_2y_1 + j^2y_1y_2 \\ &= x_1x_2 - y_1y_2 + j(x_1y_2 + x_2y_1) \end{aligned} \tag{2.8}$$

Example (Exercise Class)
Determine $z_1 \times z_2$ where $z_1 = 3 + j2$ and $z_2 = 5 + j3$
Solution

$$\begin{aligned} z_1 z_2 &= z_1 \times z_2 = (3 + j2) \times (5 + j3) && (2.9) \\ &= 15 + j9 + j10 + j^2 6 && (2.10) \\ &= 15 + j19 - 6 && (2.11) \\ &= 9 + j19 && (2.12) \end{aligned}$$

The devision of complex numbers slightly more complicated, consider the complex number

$$z = \frac{x_1 + jy_1}{x_2 + jy_2} \tag{2.13}$$

to evaluate this expression we multiply the top and bottom by $x_2 - jy_2$ giving

$$\begin{aligned} z &= \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} \\ &= \frac{x_1x_2 + y_1y_2 + j(x_2y_1 - x_1y_2)}{(x_2^2 + y_2^2)} \end{aligned} \tag{2.14}$$

the number $x - jy$ is called the **complex conjugate** of $z = x + jy$ and is denoted by z^*. Note that

$$z + z^* = 2x = 2\text{Re}(z) \tag{2.15}$$

$$z - z^* = j2y = j2\text{Im}(z) \tag{2.16}$$

and

$$zz^* = x^2 + y^2 = |z|^2 \tag{2.17}$$

where $|z|$ is the **modulus** of z.

Example (Exercise Class)
Determine $\frac{z_1}{z_2}$ where $z_1 = 1 + j4$ and $z_2 = 3 - j2$. Henceforth determine the real and imaginary parts of $\frac{z_1}{z_2}$ and $\left|\frac{z_1}{z_2}\right|$
Solution

$$\frac{z_1}{z_2} = \frac{1 + j4}{3 - j2} = \frac{(1 + j4)(3 + j2)}{(3 - j2)(3 + j2)} \tag{2.18}$$

$$= \frac{3 + j2 + j12 + j^2 8}{9 + j6 - j6 - j^2 4} \tag{2.19}$$

$$= \frac{-5 + j14}{13} \tag{2.20}$$

The real part of $\frac{z_1}{z_2}$ is $-\frac{5}{13}$ and the imaginary part is $\frac{14}{13}$. The modulus of $\frac{z_1}{z_2}$ is $\sqrt{\frac{221}{169}}$.

2.4 Graphical Representation

Complex numbers can be represented as points on a plane in a similar way to which real numbers are represented by points on a straight line. The number $z = x + jy$ is represented by the point P with coordinates (x, y). Figure 2.1 (a) shows a sequence of complex numbers and their graphical representations. Such a diagram is called an **Argand diagram** after one of its inventors. The x axis is called the **real axis** and the y axis is called the **imaginary axis**.

Following the introduction of the Argand diagram we now have another method of specifying a complex number. As indicated in Figure 2.1 (b), the point P is uniquely determined if we know the length of the line OP and the angle it makes with the x axis. The length OP is a measure of the size of z, and is called the **modulus** of z, which is usually written as denoted by mod$z$ or $|z|$. The angle between the real axis and OP is called the **argument** of z, and is denoted by arg z. Note that the polar coordinates (r, θ) and $(r, \theta + 2\pi)$ represent the same point, however, a convention is adopted to determine the argument of z uniquely: We restrict its range so that $-\pi \leq \theta \leq \pi$. The argument of the complex number $0 + j0$ is not defined.

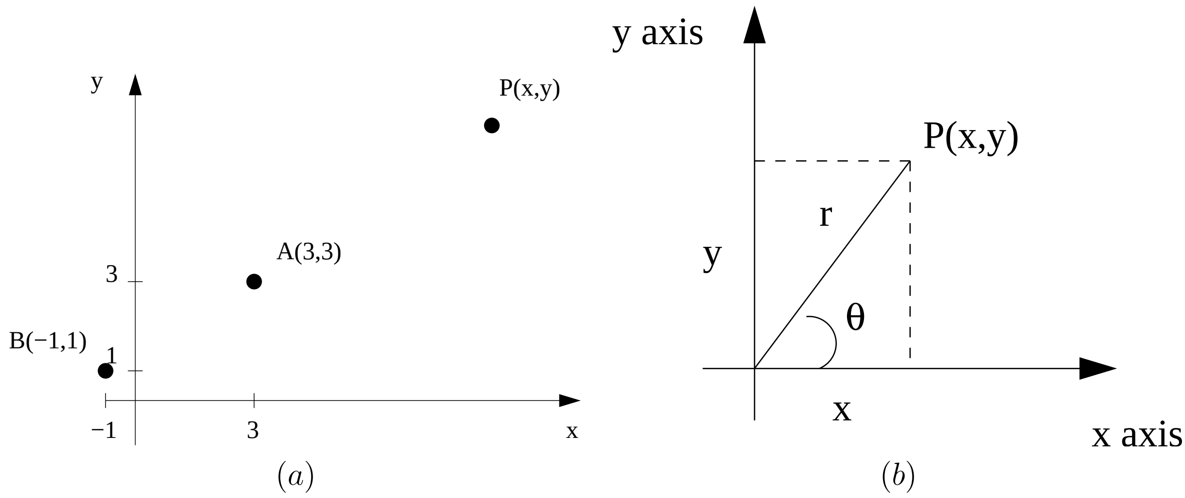


Figure 2.1: Argand diagrams of complex numbers showing, (a) a selection of complex numbers with the point A representing $3 + j3$ and B representing $-1 + j$, and (b) the polar form of the complex number $z = x + jy$

Thus from Figure 2.1 (b) $|z|$ and arg x are given by

$$|z| = r = \sqrt{x^2 + y^2} \tag{2.21}$$

$$\arg z = \theta \qquad \text{where } \tan\theta = \frac{y}{x} \tag{2.22}$$

Care must be taken to ensure that arg z is computed for the correct quadrant. By plotting the complex number in the Argand diagram one can ensure that the result makes sense.

2.5 Polar Form of a Complex Number

From Figure 2.1 (b) we easily obtain the relationships between (x, y) and (r, θ)

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \tag{2.23}$$

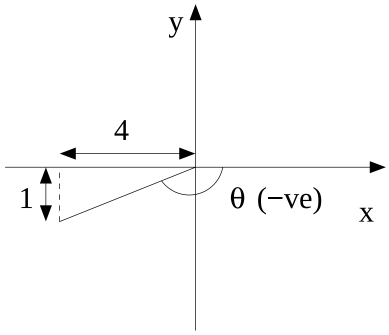
It therefore follows that the complex number $z = x + jy$ can be expressed in the form

$$z = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta) \tag{2.24}$$

This is called the **polar form** of the complex number and is frequently written as $r\angle\theta$

$$z = r\angle\theta = r(\cos\theta + j\sin\theta) \tag{2.25}$$

Example
Express $-4 - j$ in polar form.
Solution
First, we sketch the Argand diagram which shows the location of $z = -4 - j$



Thus

$$|z| = \sqrt{(-4)^2 + (-1)^2} = \sqrt{16+1} = \sqrt{17} \quad (2.26)$$
$$\arg z = -\pi + \tan^{-1}\frac{1}{4} = -2.89 \text{ (2dp) Radians} \quad (2.27)$$

Thus the polar form of the number is

$$z = \sqrt{17}(\cos -2.89 + j\sin -2.89) = \sqrt{17}(\cos 2.89 - j\sin 2.89) \quad (2.28)$$

2.5.1 Multiplication in polar form

Let $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$ then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned} \quad (2.29)$$

which by using trigonometrical identities gives

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)] \quad (2.30)$$

Hence

$$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2| \quad (2.31)$$

and

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2 \quad (2.32)$$

When using these results care must be taken to ensure that $-\pi < \arg(z_1 z_2) \leq \pi$.

Effect of multiplying by j

Since $z = r(\cos\theta + j\sin\theta)$ and j can be written as $j = 1(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2})$ it follows that

$$jz = r[\cos(\theta + \frac{\pi}{2}) + j\sin(\theta + \frac{\pi}{2})] \quad (2.33)$$

Thus the effect of multiplying a complex number by j is to leave the modulus unaltered but increase the argument by $\frac{\pi}{2}$. This property is of importance in the application of complex numbers to the theory of alternating current.

Example
For $z_1 = 2 + j3$ and $z_2 = 3 - j2$ determine $|z_1 z_2|$ and arg $(z_1 z_2)$
Solution
For z_1 we have

$$|z_1| = \sqrt{2^2 + 3^2} = \sqrt{13} \tag{2.34}$$

$$\arg z_1 = \tan^{-1}\frac{3}{2} = 0.982(3dp) \tag{2.35}$$

and for z_2 we have

$$|z_2| = \sqrt{3^2 + (-2)^2} = \sqrt{13} \tag{2.36}$$

$$\arg z_2 = -\tan^{-1}\frac{2}{3} = -0.588(3dp) \tag{2.37}$$

This means that $|z_1 z_2| = 13$ and arg $(z_1 z_2) = 0.394$

2.5.2 Division in polar form

To enable us to work out the devision of two complex numbers expressed in polar form, let us first evaluate $\frac{1}{z}$ where $z = \cos\theta + j\sin\theta$ then

$$\frac{1}{\cos\theta + j\sin\theta} = \frac{1}{\cos\theta + j\sin\theta}\frac{\cos\theta - j\sin\theta}{\cos\theta - j\sin\theta} \tag{2.38}$$

$$= \frac{\cos\theta - j\sin\theta}{\cos^2\theta + \sin^2\theta} \tag{2.39}$$

$$= \cos\theta - j\sin\theta \qquad \text{since} \cos^2\theta + \sin^2\theta = 1 \tag{2.40}$$

Using this result we can now work out $\frac{z_1}{z_2}$ where $z_1 = r_1(\cos\theta_1 + j\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)} \tag{2.41}$$

$$= \frac{r_1}{r_2}(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 - j\sin\theta_2) \tag{2.42}$$

$$= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)] \tag{2.43}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)] \tag{2.44}$$

where a trigonometric identity was used in the final step. Hence

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \tag{2.45}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2 \tag{2.46}$$

Example (Exercise Class)
For $z_1 = 2 - j4$ and $z_2 = 3 + j2$ determine $\left|\frac{z_1}{z_2}\right|$ and arg $\left(\frac{z_1}{z_2}\right)$
Solution
For z_1 we have

$$|z_1| = \sqrt{2^2 + (-4)^2} = \sqrt{20} \tag{2.47}$$
$$\arg z_1 = -\tan^{-1}\frac{4}{2} = -1.107(3dp) \tag{2.48}$$

and for z_2 we have

$$|z_2| = \sqrt{3^2 + (2)^2} = \sqrt{13} \tag{2.49}$$
$$\arg z_2 = \tan^{-1}\frac{2}{3} = 0.588(3dp) \tag{2.50}$$

This means that $\left|\frac{z_1}{z_2}\right| = \sqrt{20/13}$ and arg $\left(\frac{z_1}{z_2}\right) = -1.695$

2.6 Euler's Formula

We have defined the exponential function e^x for real values of x. We now wish to extend this definition for complex numbers. Setting $z = x + jy$ we have

$$e^z = e^{x+jy} = e^x e^{jy} \tag{2.51}$$

in the above e^x is already defined since x is a real number. We now have to define e^{jy} where y is a real number. But in general e^{jy} will be a complex number. We can easily find its modulus since we know that $zz^* = |z|^2$ and $(e^{jy})^* = e^{-jy}$

$$|e^{jy}|^2 = e^{jy}e^{-jy} = e^0 = 1 \tag{2.52}$$

Thus

$$e^{jy} = \cos\theta + j\sin\theta \qquad \text{for some } \theta \tag{2.53}$$

We now have to find the relationship between y and θ. We know that when we multiply two complex numbers together we have

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)] \tag{2.54}$$

Putting $z_1 = e^{jy_1}$ and $z_2 = e^{jy_2}$, we have

$$e^{jy_1}e^{jy_2} = e^{j(y_1+y_2)} \tag{2.55}$$
$$= \cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2) \tag{2.56}$$
$$\tag{2.57}$$

so we can see that there is a linear relationship between y and θ. Since $\theta = 0$ when $y = 0$, we deduce that $\theta \propto y$ and by remembering the identity $e^x = \cosh x + \sinh x$ we conclude that the constant of proportionality is one. Thus we deduce that

$$e^{jy} = \cos y + j\sin y \tag{2.58}$$

and this relationship is known as **Euler's formula**. Using Euler's formula enables us to write the polar form of a complex number very concisely

$$z = r(\cos\theta + j\sin\theta) = re^{j\theta} \tag{2.59}$$

This is known as the **exponential form** of the complex number.

2.6.1 Application of Euler's formula

Circular and hyperbolic functions

Euler's formula provides a theoretical link between circular and hyperbolic functions. Since

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad e^{-j\theta} = \cos\theta - j\sin\theta \tag{2.60}$$

we can deduce that

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{2.61}$$

We have previously seen that the hyperbolic functions are defined by

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2} \tag{2.62}$$

Using these results we have

$$\cosh jx = \frac{e^{jx} + e^{-jx}}{2} = \cos x \qquad \sinh jx = \frac{e^{jx} - e^{-jx}}{2} = j\sin x \tag{2.63}$$

so that

$$\tanh jx = \frac{\sinh jx}{\cosh jx} = j\frac{\sin x}{\cos x} = j\tan x \tag{2.64}$$

Also

$$\cos jx = \frac{e^{j^2x} + e^{-j^2x}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x \qquad \sin jx = \frac{e^{j^2x} - e^{-j^2x}}{2j} = \frac{e^{-x} - e^{x}}{2j} = j\sinh x \tag{2.65}$$

so that

$$\tan jx = \frac{\sin jx}{\cos jx} = j\frac{\sinh x}{\cosh x} = j\tanh x \tag{2.66}$$

Example

Find the value of $\sin[\frac{\pi}{4}(1+j)]$

Solution

We can initially use the identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$ to give

$$\sin[\frac{\pi}{4}(1+j)] = \sin\frac{\pi}{4}\cos j\frac{\pi}{4} + \cos\frac{\pi}{4}\sin j\frac{\pi}{4} \tag{2.67}$$

We can directly evaluate $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \sqrt{\frac{1}{2}}$ and make use of $\cos jx = \cosh x$ and $\sin jx = j\sinh x$ to further simplify the result.

$$\begin{aligned} \sin[\frac{\pi}{4}(1+j)] &= \sqrt{\frac{1}{2}}\cos j\frac{\pi}{4} + \sqrt{\frac{1}{2}}\sin j\frac{\pi}{4} && (2.68) \\ &= \sqrt{\frac{1}{2}}\cosh\frac{\pi}{4} + j\sqrt{\frac{1}{2}}\sinh\frac{\pi}{4} && (2.69) \\ &= 0.937 + j0.614 && (2.70) \end{aligned}$$

Logarithms

We now consider the logarithm of a complex number. If $w = \ln z$ then it is clear that $z = e^w$. Writing $z = x + jy$ and $w = u + jv$ we have

$$x + jy = e^{u+jv} = e^u e^{jv} = e^u(\cos v + j\sin v) \tag{2.71}$$

by Euler's formula. By equating real and imaginary parts we have

$$x = e^u \cos v \qquad y = e^u \sin v \tag{2.72}$$

Now squaring and adding both these equations gives

$$|z|^2 = x^2 + y^2 = e^{2u}(\cos^2 v + \sin^2 v) = e^{2u} \tag{2.73}$$

so that

$$u = \frac{1}{2}\ln(x^2 + y^2) = \ln|z| \tag{2.74}$$

To obtain v, we first divide the equation for the imaginary part by the equation for the real part to obtain

$$\tan v = \frac{y}{x} \tag{2.75}$$

so that

$$v = \text{arg}\, z \tag{2.76}$$

Finally we have

$$\ln z = \ln|z| + j\text{arg}\, z \tag{2.77}$$

Example (Exercise Class)
Evaluate $\ln(3 + j4)$ in the form $x + jy$
Solution

$$|3 + j4| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \tag{2.78}$$

$$\text{arg}\,(3 + j4) = \tan^{-1}\frac{4}{3} = 0.927 \tag{2.79}$$

so that

$$\ln(3 + j4) = \ln 5 + j0.927 = 1.609 + j0.927 \tag{2.80}$$

Powers of sine and cosine

Euler's formula may also be used to express $\sin^n\theta$ and $\cos^n\theta$ in terms of multiple angles. If $z = \cos\theta + j\sin\theta$

$$z^n = \cos n\theta + j\sin n\theta \qquad z^{-n} = \cos n\theta - j\sin n\theta \tag{2.81}$$

so that

$$z^n + z^{-n} = 2\cos n\theta \qquad z^n - z^{-n} = 2j\sin n\theta \tag{2.82}$$

Example
Express $\cos^5\theta$ in terms of multiple angles
Solution
We can use (2.81) with $n = 1$ to obtain

$$(2\cos\theta)^5 = \left(z + \frac{1}{z}\right)^5 = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5} \tag{2.83}$$

so that

$$32\cos^5\theta = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right) \tag{2.84}$$

and finally using equation (2.81) again but now with $n = 5, 3, 1$ we have

$$\cos^5\theta = \frac{1}{32}(2\cos 5\theta + 10\cos 3\theta + 20\cos\theta) \tag{2.85}$$

2.7 De Moivre's Theorem

We have seen that a complex number may be expressed in terms of its modulus r and argument θ in the exponential form $z = re^{j\theta}$. Using the rules of indices and a property of the exponential function we have, for any n

$$z^n = r^n(e^{j\theta})^n = r^n e^{j(n\theta)} \tag{2.86}$$

so that

$$z^n = r^n(\cos n\theta + j\sin n\theta) \tag{2.87}$$

This result is known as **de Moivre's theorem**.

Example
Express $1 + j$ in the form $r(\cos\theta + j\sin\theta)$ and hence find $(1 + j)^4$
Solution

$$|1 + j| = \sqrt{1^2 + 1^2} = \sqrt{2} \tag{2.88}$$

$$\arg(1 + j) = \tan^{-1}\frac{1}{1} = \frac{\pi}{4} \tag{2.89}$$

so that $1 + j = \sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$ which means that

$$(1 + j)^4 = (\sqrt{2})^4\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)^4 \tag{2.90}$$

Now using de Moivre's theorem we have

$$(1 + j)^4 = (\sqrt{2})^4\left(\cos 4\frac{\pi}{4} + j\sin 4\frac{\pi}{4}\right) \tag{2.91}$$

$$= 4(-1 + j0) \tag{2.92}$$

$$= -4 \tag{2.93}$$

2.8 Summary and Further Reading

The key topics from this chapter and references to further reading in James (fourth edition) and Croft and Davison (second edition) are

- The complex number $a + jb$ Croft and Davison [pg 424-429]. James [pg 185-186]
- Manipulation of complex numbers. Croft and Davison [pg 430-434]. James [pg 187-191]
- Argand diagram. Croft and Davison [pg 435-443]. James [pg 191-196].
- Polar from. Croft and Davison [pg. 443-449]. James [196-200].
- Euler's formula. Croft and Davison [pg 450-455]. James [pg. 200-208].
- De Moivre's theorem. Croft and Davison [pg 456-461]. James [pg. 208-215].

Chapter 3

Differential Equations

In this chapter we shall discuss differential equations. A **differential equation** is an equation which involves derivatives, an equation which contains integrals is an **integral equation** and an equation containing both derivatives and integrals is an **integro–differential equation**. Our interest lies here with trying to find the solution to certain classifications of differential equations.

Indeed, as we will see shortly, we already have the tools at our disposal to solve certain differential equations. Lets begin by talking about some of the classifications of differential equations.

3.1 Classification of differential equations

It turns out that there are many different methods to solve differential equations. So that we know what method we should apply to solve a particular differential equation it is useful to classify the different types of differential equation.

3.1.1 Ordinary and partial differential equations

Differential equations may either involve normal or partial derivatives. Differential equations which only involve normal derivatives are called **ordinary differential equations** and are sometimes abbreviated as ODE's, differential equations which involve partial derivatives are called **partial differential equations** or PDE's for short. Thus

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 2x^2 + 4y$$

is an example of a partial differential equation where as

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}f}{\mathrm{d}x} = \cos 2x$$

is an example of an ordinary differential equation.

Here we restrict consideration to ordinary differential equations. Partial differential equations will be dealt with in later courses.

3.1.2 Independent and dependent variables

The variables to which differentiation occurs are called the **independent variables** while those which are being differentiated are called the **dependent variables**. These names reflect that a differential equation expresses the way in which the dependent variable (or variables) depend on the independent variable (or variables). Ordinary differential equations have only a single independent variable where as partial differential equations have two or dependent variables. A single ordinary differential equation usually consists of one dependent variable and one independent variable.

In the ordinary differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}f}{\mathrm{d}x} = \sin 2x$$

the independent variable is x and the dependent variable is f. The two ordinary differential equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} - 2x + 3y &= \cosh t\\ 2\frac{\mathrm{d}x}{\mathrm{d}t} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 5x + 2y &= \sinh t\end{aligned}$$

are coupled and the independent variable is t and the dependent variables are x and y.

3.1.3 Order of a differential equation

To further classify a differential equation we often talk about its order. The **order of a differential equation** is the degree of the highest derivative in the differential equation. Thus

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}f}{\mathrm{d}x} = \sin 2x$$

is a second order differential equation. The coupled differential equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} - 2x + 3y &= \cosh t\\ 2\frac{\mathrm{d}x}{\mathrm{d}t} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 5x + 2y &= \sinh t\end{aligned}$$

are both first order. Also the differential equation

$$\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 4\frac{\mathrm{d}x}{\mathrm{d}y} = 0$$

is first order despite the term $\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2$.

3.1.4 Linear and non–linear equations

Linear differential equations are those in which the dependent variable (or variables) and their derivatives do not occur as products, raised to powers or in non–linear functions. **Nonlinear equations** are those which are not linear.

The coupled equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}y}{\mathrm{d}t} - 2x + 3y &= \cosh t\\ 2\frac{\mathrm{d}x}{\mathrm{d}t} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 5x + 2y &= \sinh t\end{aligned}$$

are examples of linear differential equations. Whereas

$$
\begin{aligned}
\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 4\frac{\mathrm{d}x}{\mathrm{d}y} &= 0 \\
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x\frac{\mathrm{d}x}{\mathrm{d}t} &= 4\sin t \\
4\frac{\mathrm{d}x}{\mathrm{d}t} + \sin x &= 0
\end{aligned}
$$

are all non–linear differential equations.

3.1.5 Homogeneous and non–homogeneous equations

Note that in all the examples we have presented so far all the terms involving the dependent variable appear on the left hand side of the equation and all those involving the independent variable appear on the right. When *linear equations* are arranged in this way and the right hand side is zero we call it a **homogeneous differential equation**. And when the right hand side of a linear equation is not zero we call it a **non–homogeneous differential equation**. Thus the equations

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} + 4x &= 0 \\
4\frac{\mathrm{d}x}{\mathrm{d}t} + x\sin t &= 0
\end{aligned}
$$

are homogeneous differential equations. The equations

$$
\begin{aligned}
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + t\frac{\mathrm{d}x}{\mathrm{d}t} &= 4\sin t \\
\frac{\mathrm{d}^2f}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}f}{\mathrm{d}x} &= \cos 2x
\end{aligned}
$$

are non–homogeneous differential equations.

3.2 First order differential equation

3.2.1 Implicit and explicit solutions

We consider first order ODE's that can be put in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{3.1}$$

here f is any function of x and y.

If we are able to obtain a solution to this equation that can be written in the form $y = \Phi(x)$ which satisfies (3.1) on the (possibly infinite) interval I with

$$I = (a, b) = \{x : x \in \mathbb{R}, a < x < b\} \tag{3.2}$$

We call $y = \Phi(x)$ an **explicit** solution to (3.1). Note that this implies that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \Phi'(x) = f(x, \Phi(x)) \qquad \text{for all } x \in I \tag{3.3}$$

On the otherhand, if we obtain a solution of the form

$$G(x, y) = 0$$

which, when definited implicitly satifies (3.1) we call it an **implicit solution** to (3.1).

In what follows we shall only be interested in obtaining implicit solutions to ODE's.

Example

Show that $x + y + e^{xy} = 0$ is an implicit solution to the differential equation

$$(1 + xe^{xy})\frac{\mathrm{d}y}{\mathrm{d}x} + 1 + ye^{xy} = 0$$

Solution

Differentiating $x + y + e^{xy} = 0$ with respect to x gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(x + y + e^{xy}\right) &= 0 \quad (3.4) \\ 1 + \frac{\mathrm{d}y}{\mathrm{d}x} + e^{xy}\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0 \\ (1 + xe^{xy})\frac{\mathrm{d}y}{\mathrm{d}x} + 1 + ye^{xy} &= 0 \quad (3.5) \end{aligned}$$

and so $x + y + e^{xy} = 0$ is an implicit solution to the differential equation.

3.2.2 General and Particular Solutions

Some differential equations have infinitely many solutions. For example, as we will see shortly the differential equations $\frac{\mathrm{d}y}{\mathrm{d}x} = y$ has infinitely many solutions of $y = Ae^x$ where A is any real constant. We say that this is the **general solution of the differential solution**. The general solution of a first order differential equation has one arbitrary constant. A solution to an ODE that has no arbitrary constants is called a **particular solution**. The particular solution is generally given by numerical values to the constants in the general solution.

3.2.3 Boundary and initial conditions

To obtain the particular solution to a first order differential equation (ie to ensure that it has just one solution), we have to specify a **boundary condition**. The boundary conditions specify a value of the dependent variable at a particular value of the independent variable. In the special case where all boundary conditions are given at the same value of the independent variable the boundary conditions are called **initial conditions**.

A differential equation together with its boundary conditions is called a **boundary value problem**. A differential equation together with its initial conditions is called a **initial value problem**.

Example

The differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y$ has the general solution $y = Ae^x$. Work out the particular solution for the case when $y(0) = 1$.

Solution

By substituting $y(0) = Ae^0 = 1$ we find that $A = 1$ and thus have the solution $y = e^x$

3.2.4 Variable Separable Type

ODE's of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(y)h(x) \tag{3.6}$$

are called **variable separable type** differential equations. This means that $f(x, y)$ can be written as $f(x, y) = g(y)h(x)$, ie a function of y times a function of x. Note that not every function can be written in this way (eg $f(x, y) = 1 + xy$).

To solve variable separable type differential equations, assuming that $g(y) \neq 0$ we write

$$\frac{\mathrm{d}y}{g(y)} = h(x)\mathrm{d}x$$

so that the terms on the right hand side of the equation are involving y and those on the left just involve x. Next we integrate to get the general solution.

$$\int \frac{\mathrm{d}y}{g(y)} = \int h(x)\mathrm{d}x + A$$

Example
Determine the general solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y$
Solution
We first assume $y \neq 0$ and write $\frac{\mathrm{d}y}{y} = \mathrm{d}x$ and integrating we get

$$\begin{aligned} \int \frac{\mathrm{d}y}{y} &= \int \mathrm{d}x \\ \ln|y| &= x + B \\ |y| &= e^{x+B} = Ce^x \end{aligned}$$

where C is a positive constant. Now $|y|$ could mean either y or $-y$ so that

$$\begin{aligned} y &= \pm Ce^x \\ y &= Ae^x \end{aligned}$$

where A is a nonzero constant. But, $y = 0$ is also a solution and so $y = De^x$ with D any real constant.

Example
Determine the general solution to the differential equation $\frac{dy}{dx} = \frac{y-1}{x+3}$. Find the particular solution for which $y(0) = -1$
Solution
We first assume $y - 1 \neq 0$ and write $\frac{dy}{y-1} = \frac{dx}{x+3}$ integrating gives

$$\begin{aligned} \int \frac{dy}{y-1} &= \int \frac{dx}{x+3} \\ \ln|y-1| &= \ln|x+3| + A \\ |y-1| &= e^{\ln|x+3|+A} = e^A|x+3| \\ |y-1| &= B|x+3| \end{aligned} \tag{3.7}$$

where B is a positive constant. We remember that $|X| = |Y|$ implies that $X = Y$ or $X = -Y$ so that the general solution is

$$\begin{aligned} y - 1 &= \pm B(x+3) \\ y - 1 &= C(x+3) \end{aligned}$$

where C is nonzero constant. The particular solution for $y(0) = -1$ gives $C = -\frac{2}{3}$ and $y - 1 = -\frac{2}{3}(x+3)$

3.2.5 Separable after substitution type

Some first order differential equations are not directly separable but become separable after making a simple substitution. Any first order differential equation that can be put in the form

$$\frac{dy}{dx} = k\left(\frac{y}{x}\right)$$

where $k(\cdot)$ is a function of a single variable is differential equation of this type.

Then if we put $v = \frac{y}{x}$ where v is a function of x we obtain an ODE that is satisfied by v and x and can be solved for v and hence for y. To see this, if we set $y = vx$ then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

by the product rule and thus we can write

$$\frac{dy}{dx} = k\left(\frac{y}{x}\right) = k(v) = v + x\frac{dv}{dx}$$

so that we have

$$\frac{dv}{dx} = \frac{k(v) - v}{x}$$

which is of the separable type with general solution

$$\int \frac{dv}{k(v) - v} = \int \frac{dx}{x} + A$$

After integrating we replace v by y/x.

Example
Find the general solution to the first order ODE

$$3xy^2\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + y^3$$

Solution
We observe that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3+y^3}{3xy^2}$ is not of the separable type. If we divide the top and bottom by x^3 we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+(y/x)^3}{3(y/x)^2} = k(y/x) \qquad \text{where} k(t) = \frac{1+t^3}{3t^2}$$

This is the separable after substitution type so we let $v = y/x$ or $y = vx$. Thus

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1+v^3}{3v^2} \\ x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1+v^3}{3v^2} - v = \frac{1-2v^3}{3v^2}\end{aligned}$$

and we have separable differential equation $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}\frac{1-2v^3}{3v^2}$ and if we that assume $\frac{1-2v^3}{3v^2} \neq 0$ we get

$$\begin{aligned}\int \frac{3v^2}{1-2v^3}\mathrm{d}v &= \int \frac{\mathrm{d}x}{x} \\ -\frac{1}{2}\ln|1-2v^3| &= \ln|x| + A \\ -A &= \ln(|x|\sqrt{|1-2v^3|}) \\ e^{-A} = B &= |x|\sqrt{|1-2v^3|}\end{aligned}$$

where B is any real non–zero constant, inserting $v = y/x$ and squaring gives

$$x^2\frac{|x^3-2y^3|}{|x^3|} = B^2 \qquad |x^3-2y^3| = B^2|x|$$

Thus we have

$$(x^3-2y^3) = \pm B^2 x \qquad (x^3-2y^3) = Cx$$

where C is any non zero constant. If we substitute $C = 0$ we get

$$x^3 - 2y^3 = 0 \qquad x^3 = 2y^3 \qquad \frac{1}{2} = \left(\frac{y}{x}\right)^3 = v^3$$

thus $v^3 = 1/2$. In turns out that this is indeed a solution to the differential equation $\frac{\mathrm{d}v}{\mathrm{d}x}$ and is exactly the solution for which $\frac{1-2v^3}{3v^2} = 0$. Thus the general solution is

$$(x^3-2y^3) = Cx$$

with C any real number.

3.2.6 Linear Type

Most general first order **linear type** ODE's are of the form

$$R(x)\frac{\mathrm{d}y}{\mathrm{d}x} + S(x)y = T(x)$$

where $R(x)$, $S(x)$ and $T(x)$ are given functions of x. Note that if $T(x) = 0$ then the ODE is of the separable type already discussed. For cases when $T(x) \neq 0$ then we put the equation into standard form by dividing by $R(x)$ to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} + N(x)y = M(x)$$

where $N(x) = S(x)/R(x)$ and $M(x) = T(x)/R(x)$.

To solve this type of ODE we multiply the equation by $e^{\int N(x)\mathrm{d}x}$. This is called the **integrating factor** of the ODE and gives

$$e^{\int N(x)\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + yN(x)e^{\int N(x)\mathrm{d}x} = M(x)e^{\int N(x)\mathrm{d}x}$$

Now

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\int N(x)\mathrm{d}x}y\right) &= e^{\int N(x)\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\int N(x)\mathrm{d}x}\right)\\ &= e^{\int N(x)\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + yN(x)e^{\int N(x)\mathrm{d}x}\end{aligned}$$

So that we have the ODE

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\int N(x)\mathrm{d}x}y\right) = M(x)e^{\int N(x)\mathrm{d}x}$$

When we integrate this equation we get

$$e^{\int N(x)\mathrm{d}x}y = \int\left(M(x)e^{\int N(x)\mathrm{d}x}\right)\mathrm{d}x + A$$

Hence the general solution to the equation is

$$y = e^{-\int N(x)\mathrm{d}x}\left[\int\left(M(x)e^{\int N(x)\mathrm{d}x}\right)\mathrm{d}x\right] + Ae^{-\int N(x)\mathrm{d}x}$$

Example
Find the general solution to the ODE

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = \frac{x^4}{\sqrt{1+x^3}} \qquad \text{with } x > 0$$

and find the particular solution that satisfies $y(2) = 6$.
Solution
We first write the equation in standard form

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y = \frac{x^3}{\sqrt{1+x^3}}$$

In this case $N(x) = -1/x$ and the integrating factor is

$$e^{-\int \frac{1}{x}\mathrm{d}x} = e^{-\ln|x|} = e^{\ln|x|^{-1}} = e^{\ln\frac{1}{|x|}} = \frac{1}{|x|} = \frac{1}{x}$$

since $x > 0$. If we multiply the differential equation by the integrating factor we get

$$\begin{aligned} \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x^2}y &= \frac{x^2}{\sqrt{1+x^3}} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}y\right) &= \frac{x^2}{\sqrt{1+x^3}} \end{aligned}$$

If we integrate both sides we get

$$\begin{aligned} \frac{1}{x}y &= \int \frac{x^2}{\sqrt{1+x^3}}\mathrm{d}x + A \\ \frac{1}{x}y &= \frac{2}{3}\sqrt{1+x^3} + A \\ y &= \frac{2}{3}x\sqrt{1+x^3} + Ax \end{aligned}$$

where A is any real constant. The particular solution for which $y(2) = 6$ gives

$$6 = \frac{2}{3}\cdot 2 \cdot \sqrt{9} + 2A \qquad \text{hence } A = 1$$

Thus

$$y = x\left(\frac{2}{3}\sqrt{1+x^3} + 1\right)$$

3.2.7 More specialised types

Type A

We can solve ODE's of the type

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(ax + by)$$

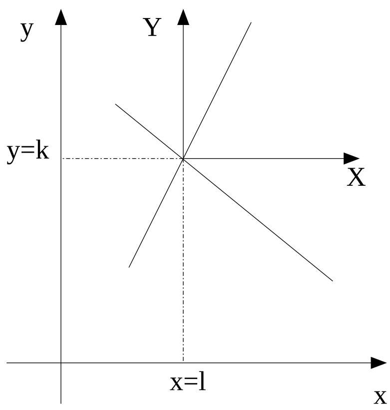


Figure 3.1: Change of axis to (X, Y)

where a and b are known constants and g is a known function. By using the substitution$z = ax + by$ we have that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = a + b\frac{\mathrm{d}y}{\mathrm{d}x} \tag{3.8}$$

So that we can write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{b}\left(\frac{\mathrm{d}z}{\mathrm{d}x} - a\right) = g(z) \qquad \text{and } \frac{\mathrm{d}z}{\mathrm{d}x} = a + bg(z)$$

This equation is now of a separable type in z and x and has solution

$$\int \frac{\mathrm{d}z}{a + bg(z)} = \int \mathrm{d}x + A$$

Type B

An ODE of the type

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax + by + e}{cx + fy + g}$$

is not separable but we can make a simple substitution to make it the same as in type A or separable after a substitution so that we can solve it. Let us suppose that

$$\begin{aligned} ax + by + e &= 0 \\ cx + fy + g &= 0 \end{aligned}$$

represent two lines. If these two lines are parallel then $cx + fy = \lambda(ax + by)$ for some constant λ. If λ exists then we can write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax + by + e}{\lambda(ax + by) + g} = h(ax + by) \qquad \text{where } h(t) = \frac{t + e}{\lambda t + g}$$

and thus we have an equation which is of type A.

If the two lines are not parallel, they intersect say at the point (ℓ, k). We change the coordinate axis from (x, y) to (X, Y) as shown in Figure 3.1, so the position of the new origin in the old coordinate system is (ℓ, k). Thus

$$X = x - \ell \qquad Y = y - k$$

The equation of the 2 straight lines in the (X, Y) coordinate system are

$$\begin{aligned} aX + bY &= 0 \\ cX + fY &= 0 \end{aligned}$$

and therefore the ODE can be re–written as

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{aX + bY}{cX + fY}$$

which is of the separable type after substitution and can be solved by putting $v = Y/X$

Type C

The final first order type that we wish to explore are ODE's of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

when $n \neq 0$. The is also known as **Bernoulli's equation**. We note that when $n = 0$ then this is a first order linear equation and when $n = 1$ then the equation is a first order linear separable equation.

To solve this type of equation we assume that $y \neq 0$ and divide both sides by y^n to get

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x)$$

If we set $z = y^{1-n}$ then $\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$. Thus the differential equation becomes

$$\frac{1}{1-n}\frac{\mathrm{d}z}{\mathrm{d}x} + P(x)z = Q(x)$$

which is a first order linear ODE in Z and x. If we express it in standard form we have

$$\frac{\mathrm{d}z}{\mathrm{d}x} + (1-n)P(x)z = (1-n)Q(x)$$

and it has integrating factor $e^{\int (1-n)P(x)\mathrm{d}x}$. Thus we can solve for z and substitute back to get y.

Example
Find the general solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = -\frac{5}{2}xy^3$$

Solution
We recognise it as an example of the Bernoulli equation. Assuming that $y \neq 0$ and dividing by y^3 we get

$$y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} - 5y^{-2} = -\frac{5}{2}x$$

If we let $z = y^{-2}$ then $\frac{\mathrm{d}z}{\mathrm{d}x} = -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x}$ then the ODE becomes

$$-\frac{1}{2}\frac{\mathrm{d}z}{\mathrm{d}x} - 5z = -\frac{5}{2}x$$

which when expressed in standard form is

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 10z = 5x$$

The integrating factor is $e^{\int 10\mathrm{d}x} = e^{10x}$. Multiplying by the integrating factor gives

$$\begin{aligned}
e^{10x}\frac{\mathrm{d}z}{\mathrm{d}x} + 10ze^{10x} &= 5xe^{10x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{10x}z\right) &= 5xe^{10x} \\
e^{10x}z &= 5\int xe^{10x}\mathrm{d}x = 5\left[\frac{xe^{10x}}{10} - \int \frac{e^{10x}}{10}\mathrm{d}x\right] \\
&= 5\left[\frac{xe^{10x}}{10} - \frac{e^{10x}}{100}\right] + A
\end{aligned}$$

where A is an arbitrary constant of integration. We have that

$$z = \frac{x}{2} - \frac{1}{20} + Ae^{-10x} \qquad \frac{1}{y^2} = \frac{x}{2} - \frac{1}{20} + Ae^{-10x}$$

Earlier we made the assumption that $y \neq 0$ however, $y = 0$ also satisfies the ODE so it is also a solution.

3.3 Second Order ODE's

For simplicity we shall only consider second order linear ODE's. These are equations of the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = f(x) \tag{3.9}$$

here $P(x)$, $Q(x)$ and $f(x)$ are all given continuous functions.

3.3.1 Homogeneous equations

For the special case where $f(x) = 0$ in (3.9) then

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = 0 \tag{3.10}$$

and this is called a second order **homogeneous differential equation**. The general solution to equations of this form is

$$y(x) = A_1 y_1(x) + A_2 y_2(x) \tag{3.11}$$

where A_1 and A_2 are arbitrary constants and $y_1(x)$ and $y_2(x)$ are any linearly independent solutions to (3.10). We first came across linear independence when talking about vectors, when applied to scalers this means that $y_1(x)$ is not a multiple of $y_2(x)$.

3.3.2 Linear equations

We can use the general solution to (3.10) to get the general solution to (3.9). All we need to find is a particular solution $y = p(x)$ to (3.9) then the general solution to (3.9) is given by

$$y(x) = A_1 y_1(x) + A_2 y_2(x) + p(x) \tag{3.12}$$

in other words the general solution to a second order ODE is the general solution to (3.10) plus a particular solution to (3.9).

We can check this as follows, if $y = A_1 y_1 + A_2 y_2 + p$ then

$$y' = A_1 y_1' + A_2 y_2' + p' \qquad y'' = A_1 y_1'' + A_2 y_2'' + p''$$

We note that $y_1'' + P(x)y_1' + Q(x)y_1' = 0$, $y_2'' + P(x)y_2' + Q(x)y_2' = 0$, $Q(x)y = (A_1 y_1 + A_2 y_2 + p)Q(x)$ and $P(x)y' = (A_1 y_1' + A_2 y_2' + p')P(x)$. Thus we have

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= [A_1 y_1'' + A_2 y_2'' + p''] + P(x)[A_1 y_1' + A_2 y_2' + p'] + Q(x)[A_1 y_1 + A_2 y_2 + p] \\ &= p'' + P(x)p' + Q(x)p = f(x) \end{aligned}$$

Hence the general solution to (3.9) is

$$y(x) = \text{general solution of the homogeneous equation} + \text{any particular solution to (3.9)}$$

We call the general solution to the homogeneous equation the **complementary function** and the particular solution the **particular integral**.

3.3.3 Linear equations with constant coefficients

We restrict consideration to equations of the type

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x) \tag{3.13}$$

where a_1 and a_0 are constants and $f(x)$ is continuous. To obtain the solution we need to find the complementary function and the particular solution.

To find the complementary function

This is the solution to the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0y = 0 \tag{3.14}$$

to find the solution to this equation, we first write down the polynomial equation

$$m^2 + a_1m + a_0 = 0 \tag{3.15}$$

this is an equation in m and is called the **auxiliary equation**. We find the roots of this equation $m = m_1, m_2$ say. The general solution to (3.14) then are

- If the roots m_1 and m_2 are real and m_1 is different from m_2 then the the complementary function is of the form
$$y = A_1e^{m_1x} + A_2e^{m_2x}$$
where A_1, A_2 are arbitrary constants
- If the roots are equal $m_1 = m_2$ then the complementary function is given by
$$y = (A_1 + A_2x)e^{m_1x}$$
where A_1, A_2 are arbitrary constants
- If the roots are complex then m_1 and m_2 are complex conjugates say $p \pm jq$ then the complementary function is
$$y = e^{px}(A_1\cos qx + A_2\sin qx)$$
where A_1, A_2 are arbitrary constants

Example
Find the general solution to the ODE

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 10x = 0$$

Solution
We first write down the auxiliary equation

$$m^2 + 2m + 10 = 0$$

which has roots

$$m = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm j3$$

Thus the general solution is $x = e^{-t}(A_1\cos 3t + A_2\sin 3t)$ and A_1, A_2 are real constants.

To find the particular integral

To find a solution to the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0y = f(x) \tag{3.16}$$

we use the trial method. In this method we make a guess depending on the form of $f(x)$ and the complementary function. Then we substitute our guess into the ODE.

- Suppose that $f(x) = Ae^{kx}$ where A and k are given constants.
  - If k is not a root of the auxiliary equation, try $y(x) = ae^{kx}$
  - If k is a simple root of the auxiliary equation, try $y(x) = axe^{kx}$
  - If k is a double root of the auxiliary equation, try $y(x) = ax^2e^{kx}$

Example
Find the general solution to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x} = 5e^x$$

Solution
To find the complementary function, we first find the roots of the auxiliary equation $m^2 - m = m(m-1) = 0$. Hence the roots are $m = 0$ and $m = 1$. Thus the complementary function is

$$y_{cf} = A_1e^{0x} + A_2e^{1x} = A_1 + A_2e^x$$

where A_1 and A_2 are arbitrary constants. Since $m = 1$ is a simple root of the auxiliary equation we try $y = axe^x$. Thus

$$y' = ae^x(1+x) \qquad y'' = ae^x(2+x)$$

and so y is a solution provided that

$$\begin{aligned} ae^x(2+x) - ae^x(1+x) &= 5e^x \\ ae^x &= 5e^x \\ a &= 5 \end{aligned}$$

Thus $y = 5xe^x$ is a solution to $\frac{\mathrm{d}^2y}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x} = 5e^x$ and hence the particular integral is $y_{pi} = 5xe^x$. The general solution is

$$y = y_{cf} + y_{pi} = A_1 + A_2e^x + 5xe^x$$

- Suppose that $f(x) = p_0 + p_1x + \cdots + p_kx^k$ where $p_0, p_1, \cdots, p_k$ are given constants
  - If 0 is not a root of the auxiliary equation try $y(x) = b_0 + b_1x + \cdots + b_kx^k$
  - If 0 is a simple root of the auxiliary equation try $y(x) = x(b_0 + b_1x + \cdots + b_kx^k)$
  - If 0 is a double root of the auxiliary equation try $y(x) = x^2(b_0 + b_1x + \cdots + b_kx^k)$

  In all cases substitute $y(x)$ into $\frac{\mathrm{d}^2y}{\mathrm{d}x} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0y = f(x)$ and determine the constants $b_0, b_1, \cdots, b_k$.

Example
Find the general solution to

$$\frac{\mathrm{d}^2y}{\mathrm{d}x} - 2y = x^2 + 2 \tag{3.17}$$

Solution
For the complementary function the auxiliary equation is $m^2 - 2 = 0$ and has roots $m = \pm\sqrt{2}$. Thus the complementary function is

$$y_{cf} = A_1 e^{\sqrt{2}x} + A_2 e^{-\sqrt{2}x}$$

For the particular integral we try $y = b_0 + b_1 x + b_2 x^2$ where b_0, b_1, b_2 are constants to be found. Differentiating we have

$$y' = b_1 + 2b_2 x \qquad y'' = 2b_2$$

and substituting this into the differential equation gives

$$2b_2 - 2(b_0 + b_1 x + b_2 x^2) = x^2 + 2$$

Equating coefficients of x^2 gives $-2b_2 = 1$ so that $b_2 = -1/2$. Equating coefficients of x gives $-2b_1 = 0$ and it follows that $b_1 = 0$. Finally equating coefficients of x^0 gives $2b_2 - 2b_0 = 2$ which gives $b_0 = -3/2$. Thus the particular integral is

$$y_{pi} = -\frac{3}{2} - \frac{1}{2}x^2 \tag{3.18}$$

and the general solution is

$$y = y_{cf} + y_{pi} = A_1 e^{\sqrt{2}x} + A_2 e^{-\sqrt{2}x} - \frac{3}{2} - \frac{1}{2}x^2$$

- Suppose that $f(x) = A\sin kx + B\cos kx$ where A, B and k are given constants.
  - If $\sin kx$ is not a term in the complementary function, try $y = a\cos kx + b\sin kx$
  - If $\sin kx$ is a term in the complementary function try $y = x(a\cos kx + b\sin kx)$

  In all cases $y(x)$ into $\frac{\mathrm{d}^2y}{\mathrm{d}x} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x)$ and determine the constants a, b.
- Suppose that $f(x) = f_1(x) + f_2(x)$ where y_1 is a solution of $\frac{\mathrm{d}^2y}{\mathrm{d}x} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f_1(x)$ and y_2 is a solution of $\frac{\mathrm{d}^2y}{\mathrm{d}x} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f_2(x)$. Then $y(x) = y_1(x) + y_2(x)$ is a solution of $\frac{\mathrm{d}^2y}{\mathrm{d}x} + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x)$.

3.4 Summary and Further Reading

The key topics from this chapter and references to further reading in James (fourth edition) and Croft and Davison (second edition) are

- Seperable type. Croft and Davison [pg 918-930]. James [pg 786-789]
- Seperable after subsitution. James [pg 789-791].
- Integrating factor approach for linear type. Croft and Davison [pg 931-938]. James [pg 795-799].

- Bernoulli equation. James [pg 799-802].
- Second order equations with constant coefficents. Croft and Davison [pg 950-976]. James [pg 826-839].

Chapter 4

Functions of more than one variable

Last semester we looked at differentiation and integration for functions of a single variable. We saw how we could differentiate and integrate a variety of functions and looked at their importance in engineering. However, many of the functions that we come across in engineering depend on more than one variable, for example the area of a rectangular plate of width x and breadth y is given by

$$A = xy \tag{4.1}$$

The variables x and y are clearly independent of each other, so we say that the dependent variable A is a function of the two independent variables x and y. This is expressed by writing $A = f(x, y)$ or $A(x, y)$. Let us now consider the volume of a plate given by

$$V = xyz \tag{4.2}$$

where the thickness of the plate is z. In this case V is the dependent variable and x , y and z are independent variables. We write $V = f(x, y, z)$ or $V(x, y, z)$.

In general if we have a variable t which is a function of n independent variables $x_1, x_2, x_3, \cdots, x_n$ we can express this as

$$t = f(x_1, x_2, x_3, \cdots, x_n) \tag{4.3}$$

As for functions of one variable $f(x)$ which we discussed last semester, the function of n variables has an associated **domain** in n–dimensional space, a **range** and a **rule** that assigns each n–tuple of real numbers $(x_1, x_2, x_3, \cdots, x_n)$ in the n–dimensional domain with a real number z in the range.

We do not wish to pursue deeper in to these issues as our interest here lies with the differentiation and integration of functions of more than one variable. We begin this chapter with looking at how we visualise functions of more than one variable, then we move on to the topic of partial differentiation. We finish the chapter by considering integrals of surfaces and volumes.

4.1 Visualisation of Functions of Two and Three variables

For purposes of illustration we restrict ourselves to functions of two or three independent variables. Let us consider the function

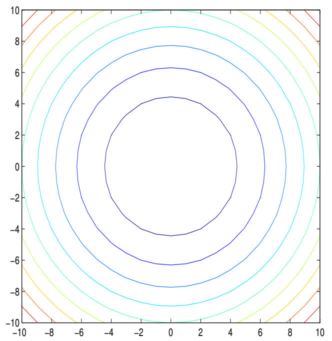
$$z = f(x, y) \tag{4.4}$$

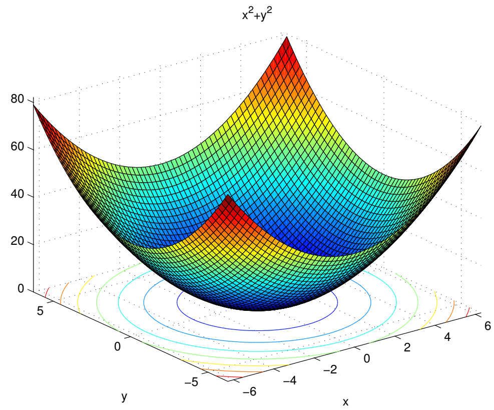
which is a function of two independent variables x and y. We have two ways of visualising such a function: The first way uses **level curves** which curves in the x, y domain on which the function $f(x, y)$ has a constant value. Level curves follow the same ideas as contours which are used to show elevation on a ordnance survey map. The second alternative is to plot the points corresponding to (x, y, z) with $z = f(x, y)$ in a rectangular coordinate system with axis x, y, z. By doing this we end up with function being represented as a surface.

Example
We wish to visualise the function

$$z = x^2 + y^2$$

Solution
By using MATLAB, we can make level surface plots and surface plots of this functions. Illustrations of both are shown below





Note that for functions of three independent variables, eg $w = f(x, y, z)$ we cannot plot surfaces like we did for functions of two variables. We can, however, plot **level surfaces**. Level surfaces are like level curves, they represent a surface on which w is constant.

4.2 Partial Differentiation

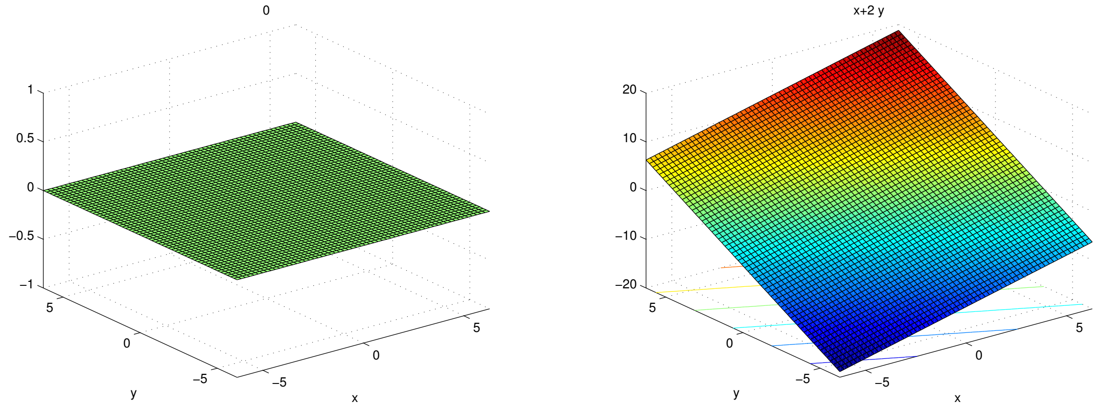
We recall from last semester that the derivative of a function $f(x)$ of one variable measures the slope of the tangent to the graph of the function. If we now consider a function $z = f(x, y)$ of two variables, slope no longer makes sense because $z = f(x, y)$ defines a surfaces in three dimension. Consider the following two cases:

- Lets start with the simplest surface $z = 0$ ie, a surface which is flat in both the x and y directions, as shown in Figure 4.1 (a). If we move along a line for which y is fixed and x is increasing, the slope of this line will be 0. Similarly if we move along a line for which x is fixed and y is increasing this line will also have zero slope.

- Next we consider the surface $z = x + 2y$, as shown in Figure 4.1 (b). For this example, the slope is equal to 1 if we move along a line of fixed y and increasing x. If, however, we move along a line for which x is fixed and y is increasing then we find that the slope is equal to 2.

It turns out that for a general surface the slope will be different depending on which direction we move in. To measure this a new kind of derivative is introduced called the **partial derivative**. Formally the partial derivative of $f(x, y)$ with respect to x is defined as

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{4.5}$$



(a) (b)

Figure 4.1: Visualisation surfaces for $z = 0$ and $z = x + 2y$

This means that we differentiate $f(x, y)$ with respect to x while keeping y constant (fixed). The partial derivative of $f(x, y)$ with respect to x is the same as measuring the slope in the x direction. We denote this partial derivative by

$$\frac{\partial f}{\partial x} \qquad \text{or} \qquad \partial f / \partial x$$

Note the use of 'curly dee's' to distinguish between partial differentiation and normal differentiation. In writing care must be taken to distinguish between

$$\frac{\mathrm{d}f}{\mathrm{d}x}, \qquad \frac{\Delta f}{\Delta x} \qquad \text{and} \qquad \frac{\partial f}{\partial x} \tag{4.6}$$

In a similar way to the partial derivative of $f(x, y)$ with respect to x, we define the partial derivative of $f(x, y)$ with respect to y as

$$\frac{\partial f}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \tag{4.7}$$

which we determine by differentiating $f(x, y)$ with respect to y by keeping x constant. This partial derivative is the same as measuring the slope in the y direction.

If we know both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ we can work out slope of the surface for any direction. If we consider a direction at an angle α to the x axis the slope is given by

$$\frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \sin \alpha \tag{4.8}$$

we call this the **directional derivative**.

Example
Given the function $f(x,y) = x^2y^3 + 3y + x$, determine its partial derivative with respect to x and y. Hence determine its directional derivative for a direction at angle α to the x axis.
Solution
To find the partial derivative of $f(x,y)$ with respect to x, we differentiate $f(x,y)$ and keep y constant. Thus

$$\frac{\partial f}{\partial x} = 2xy^3 + 1$$

Similarly, we obtain the partial derivative of $f(x,y)$ with respect to y, by differentiating $f(x,y)$ while keeping x constant

$$\frac{\partial f}{\partial y} = 3x^2y^2 + 3$$

We obtain the directional derivative by applying formula (4.8), giving

$$(2xy^3 + 1)\cos\alpha + (3x^2y^2 + 3)\sin\alpha$$

Here are some more examples

Example
Determine $\partial f/\partial x$ and $\partial f/\partial y$ when $f(x,y)$ is

$$\text{a)} \qquad x^2y^2 + 3xy - x + 2 \qquad \text{b)} \qquad \sin(x^2 - 3y)$$

Solution
a) For $f(x,y) = x^2y^2 + 3xy - x + 2$ we have

$$\frac{\partial f}{\partial x} = 2xy^2 + 3y - 1 \qquad \frac{\partial f}{\partial y} = 2x^2y + 3x$$

b) For $f(x,y) = \sin(x^2 - 3y)$ we have

$$\frac{\partial f}{\partial x} = \cos(x^2 - 3y)\frac{\partial}{\partial x}(x^2 - 3y) = 2x\cos(x^2 - 3y) \qquad \frac{\partial f}{\partial y} = -3\cos(x^2 - 3y)$$

In the examples we have considered so far we have used partial differentiation in the context of function of two variables. However, the concept may be extended to functions of as many variables as we please. For a function $f(x_1, x_2, \cdots x_n)$ of n variables, the partial derivative with respect to x_i is given by

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \cdots, x_i + \Delta x_i, x_{i+1}, \cdots, x_n) - f(x_1, x_2, \cdots, x_i, x_{i+1}, \cdots, x_n)}{\Delta x_i}$$

in practise we obtain this by differentiating the function with respect to x_i while keeping all other $n-1$ variables constant.

Example
Determine $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ when

$$f(x,y,z) = xyz^2 + 3xy - z$$

Solution
We obtain that

$$\begin{aligned} \frac{\partial f}{\partial x} &= yz^2 + 3y \\ \frac{\partial f}{\partial y} &= xz^2 + 3x \\ \frac{\partial f}{\partial x} &= 2xyz - 1 \end{aligned}$$

4.2.1 Chain rule

We already came across the chain rule when we performing standard differentiation for functions of a single variable. We now wish to extend these ideas to functions of more than one variable. Let's consider the case where $z = f(x, y)$ and x and y are themselves functions of two independent variables s and t. This means that we can also write z as a function of s and t, say $F(s, t)$. If we want to differentiate z with respect to s or t we have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{4.9}$$

We can write this in matrix notation as follows

$$\begin{pmatrix} \frac{\partial z}{\partial s} \\ \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix} \tag{4.10}$$

This result is called the **chain rule**.

Example
Find $\partial T/\partial r$ and $\partial T/\partial\theta$ when

$$T(x,y) = x^2 + 2xy + y^3x^2$$

and $x = r\cos\theta$ and $y = r\sin\theta$
Solution
By the chain rule

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r}$$

In this example

$$\frac{\partial T}{\partial x} = 2x + 2y + 2xy^3 \qquad \frac{\partial T}{\partial y} = 2x + 3x^2y^2$$

and

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

so that

$$\begin{aligned}\frac{\partial T}{\partial r} &= (2x + 2y + 2xy^3)\cos\theta + (2x + 3x^2y^2)\sin\theta \\ &= (2r\cos\theta + 2r\sin\theta + 2r^4\cos\theta\sin^3\theta)\cos\theta + (2r\cos\theta + 3r^4\cos^2\theta\sin^2\theta)\sin\theta\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial T}{\partial\theta} &= -(2x + 2y + 2xy^3)r\sin\theta + (2x + 3x^2y^2)r\cos\theta \\ &= -(2r\cos\theta + 2r\sin\theta + 2r^4\cos\theta\sin^3\theta)r\sin\theta + (2r\cos\theta + 3r^4\cos^2\theta\sin^2\theta)r\cos\theta\end{aligned}$$

Example
Find $\mathrm{d}R/\mathrm{d}s$ when

$$R(s) = \cosh(x^2 + 3y)$$

and $x = s^2 + 3s$ and $y = \sin s$
Solution
For this example, x and y are functions of s only so

$$\frac{\mathrm{d}R}{\mathrm{d}s} = \frac{\partial R}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial R}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s}$$

which gives

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}s} &= 2x(2s+3)\sinh(x^2+3y) + 3\cos s\sinh(x^2+3y) \\ &= 2(s^2+3s)(2s+3)\sinh((s^2+3s)^2 + 3\sin s) + 3\cos s\sinh((s^2+3s)^2 + 3\sin s) \\ &= 2(2s^3+9s^2+9s)\sinh((s^2+3s)^2 + 3\sin s) + 3\cos s\sinh((s^2+3s)^2 + 3\sin s)\end{aligned}$$

4.2.2 Higher order partial derivatives

So far we have considered functions like $f(x,y)$ and found its partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. If the partial derivatives are also functions of x and y, they can also be differentiated with respect to x an y. We define higher order partial derivatives as follows

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\\ \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\end{aligned}$$

If $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are continuous, then it follows that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{4.11}$$

Note, however, that if the conditions are not fulfilled these so called **mixed partial derivatives** are not equal.

Example
For the function

$$f(x, y) = \sin x \cos y + x^3 e^y \tag{4.12}$$

find all the second order partial derivatives
Solution
First we find the first order partial derivatives

$$\frac{\partial f}{\partial x} = \cos x \cos y + 3x^2 e^y \qquad \frac{\partial f}{\partial y} = -\sin x \sin y + x^3 e^y$$

Then by differentiating these expressions again we can find the second order derivatives

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= -\sin x \cos y + 6xe^y\\ \frac{\partial^2 f}{\partial y^2} &= -\sin x \cos y + x^3 e^y\\ \frac{\partial^2 f}{\partial x \partial y} &= -\cos x \sin y + 3x^2 e^y\\ \frac{\partial^2 f}{\partial y \partial x} &= -\cos x \sin y + 3x^2 e^y\end{aligned}$$

In this case, we have that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

4.2.3 Total differentiation

Let us consider the function $z = f(x, y)$ which is a function of two variables x and y. Now let Δx represent a small change in x, Δy a small change in y and Δz a small change in z.

It follows that

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

we can rewrite this as the sum of two terms, the first of which shows the change in z due to a change in x and the second which shows the change in z due to a change in y

$$\Delta z = [f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]$$

Next, we multiply the first term by $\Delta x/\Delta x = 1$ and the second term by $\Delta y/\Delta y = 1$

$$\Delta z = \frac{[f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)]}{\Delta x}\Delta x + \frac{[f(x, y + \Delta y) - f(x, y)]}{\Delta y}\Delta y$$

By letting Δx, Δy and Δz tend to zero we get

$$\mathrm{d}z = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \tag{4.13}$$

In this expression $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ are called **differentials**. If $z = f(x)$ so that it is a function of one variable, the formula takes the form

$$\mathrm{d}z = \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x \tag{4.14}$$

If $w = f(x, y, z)$ is a function of three variables we have

$$\mathrm{d}w = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z \tag{4.15}$$

We can use the idea of differentials to calculate errors. If $z = f(x, y)$ and Δx and Δy are errors in x and y, then the error in z is approximately given by

$$\Delta z \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \tag{4.16}$$

Example
We want to estimate $\sqrt{(3.01)^2 + (3.97)^2}$
Solution
Let $z = f(x, y) = \sqrt{x^2 + y^2}$. If we set $x = 3$ and $y = 4$ we can easily compute $z = \sqrt{3^2 + 4^2} = 5$. Now $\sqrt{(3.01)^2 + (3.97)^2}$ is z when x is increased by $\Delta x = 0.01$ and when y is decreased by 0.03, ie $\Delta y = -0.03$

$$\begin{aligned}
\Delta z &\approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \\
&= \frac{1}{2}2x(x^2 + y^2)^{-1/2}\Delta x + \frac{1}{2}2y(x^2 + y^2)^{-1/2}\Delta y \\
&= \frac{x}{\sqrt{x^2 + y^2}}\Delta x + \frac{y}{\sqrt{x^2 + y^2}}\Delta y \\
&= \left(\frac{3}{5} \times 0.01\right) + \frac{4}{5} \times (-0.03) = -0.018
\end{aligned}$$

So $\sqrt{(3.01)^2 + (3.97)^2} \approx 5 + \Delta z = 5 - 0.018 \approx 4.98$

Example
The height of a cylinder is under measured by 3% and the radius is over measured by 2% we wish to estimate the percentage error in the volume.
Solution
The volume of a cylinder is given by $V = \pi r^2 h$ so

$$\frac{\partial V}{\partial r} = 2\pi r h \qquad \frac{\partial V}{\partial h} = \pi r^2$$

So that the error in the volume may be written as

$$\begin{aligned} \Delta V &\approx \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h \\ &= 2\pi r h \Delta r + \pi r^2 \Delta h \end{aligned}$$

As we are interested in the percentage error, we divide this equation by V

$$\begin{aligned} \Delta V &\approx \frac{2\pi r h}{\pi r^2 h}\Delta r + \frac{\pi r^2}{\pi r^2 h}\Delta h \\ &= \frac{2\Delta r}{r} + \frac{\Delta h}{h} \end{aligned}$$

From the question we know that $\frac{\Delta r}{r} = \frac{2}{100}$ and $\frac{\Delta h}{h} = -\frac{3}{100}$ giving $\frac{\Delta V}{V} = \frac{1}{100}$. This means that the volume is overestimated by 1%.

4.3 Integration

As well as being able to differentiate multivariate functions we also need to be able to integrate them. In engineering, three types of integrals commonly occur: line integrals, surface integrals and volume integrals. In this section we shall look at how these may be performed.

4.3.1 Line integrals

Let us consider the integral

$$\int_a^b f(x, y)\mathrm{d}x \qquad \text{where } y = g(x) \tag{4.17}$$

we can perform the integration in the usual way, once we have substituted y for $g(x)$

$$\int_a^b f(x, g(x))\mathrm{d}x \tag{4.18}$$

Clearly the value of the integral depends on the function $y = f(x)$. We can interpret it as evaluating $\int_a^b f(x, y)\mathrm{d}x$ along the curve $y = g(x)$, as shown in Figure 4.2. The result of this integral is no longer the area under the curve and to distinguish it from our earlier integrals we call it a **line integral**.

This isn't the only type of line integral, other examples are

$$\int_C f(x, y)\mathrm{d}x \qquad \int_C f(x, y)\mathrm{d}s \qquad \int_C f(x, y)\mathrm{d}t \qquad \int_C [f_1(x, y)\mathrm{d}x + f_2(x, y)\mathrm{d}y]$$

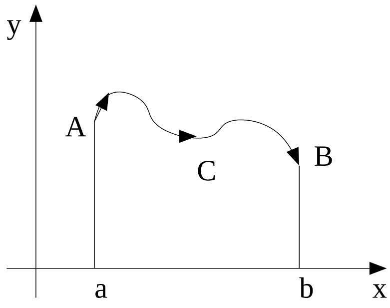
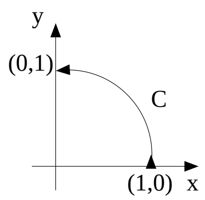


Figure 4.2: Illustration of a line integral

Note that in the above the symbol C, this means that the integral is evaluated along the curve or **path** C. The path is not restricted to two dimensions and may be in as many dimensions as we please. It is generally preferred to use C instead of the usual limits a and b when talking about line integrals, as the limits of integration are usually clear from how C is defined.

Example
Evaluate $\int_C xy\mathrm{d}x$ from $(1,0)$ to $(0,1)$ along the curve C that is the portion of $x^2+y^2=1$ in the first quadrant.



Solution
On this curve $y=\sqrt{1-x^2}$ so that

$$\int_C xy\mathrm{d}x = \int_1^0 x\sqrt{1-x^2}\mathrm{d}x = \left[-\frac{1}{2}\frac{2}{3}(1-x^2)^{3/2}\right]_1^0 = -\frac{1}{3}$$

Example
Evaluate the integral

$$I = \int_C [(x^2+2y)\mathrm{d}x + (x+y^2)\mathrm{d}y]$$

from $(0,1)$ to $(2,3)$ along the curve C defined by $y=x+1$
Solution
Since $y=x+1$ then $\mathrm{d}y=\mathrm{d}x$ and

$$\begin{aligned} I &= \int_0^2 [(x^2+2(x+1))+(x+(x+1)^2)]\mathrm{d}x \\ &= \int_0^2 (2x^2+5x+3)\mathrm{d}x = \left[\frac{2}{3}x^3+\frac{5}{2}x^2+3x\right]_0^2 = \frac{64}{3} \end{aligned}$$

Example
Evaluate
$$\int_C (z\mathrm{d}x + x^2\mathrm{d}y - 2y\mathrm{d}z)$$
along the curve C which is specified parametrically as $x = t$, $y = t^2$ and $z = t^3$ from $(0, 0, 0)$ to $(1, 1, 1)$.
Solution
On the curve C, $\mathrm{d}x = \mathrm{d}t$, $\mathrm{d}y = 2t\mathrm{d}t$ and $\mathrm{d}z = 3t^2\mathrm{d}t$. Also at the point $(0, 0, 0)$ $t = 0$ and at the point $(1, 1, 1)$ $t = 1$ so that

$$\begin{aligned}\int_C (z\mathrm{d}x + x^2\mathrm{d}y - 2y\mathrm{d}z) &= \int_0^1 (t^3\mathrm{d}t + 2t^3\mathrm{d}t - 6t^4\mathrm{d}t) \\ &= \int_0^1 (3t^3 - 6t^4)\mathrm{d}t \\ &= \left[\frac{3}{4}t^4 - \frac{6}{5}t^5\right]_0^1 = -\frac{9}{20}\end{aligned}$$

As we mentioned earlier, some line integrals may be given in the form $\int_C f(x, y)\mathrm{d}s$ where s indicates the arc length along the curve defined by $y = g(x)$. One of the simplest examples of such integrals is $\int_C \mathrm{d}s$ which is equal to the length of the curve C. To evaluate this kind integrals we note that $\mathrm{d}s$ is given by

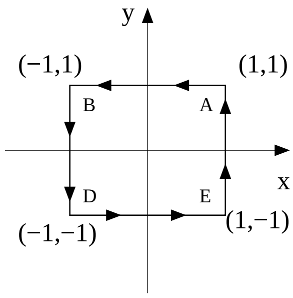
- $\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x$ in Cartesian form
- $\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\mathrm{d}t$ in parametric form
- $\mathrm{d}s = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta$ in polar form

Furthermore, if a line integral is such that the integration is performed around a closed (simple) curve, then we denote this type of integral by $\oint_C \mathrm{d}s$ with the convention that the integral is evaluated by travelling around C in an anticlockwise direction.

Example
Evaluate the integral

$$\oint_C \frac{\mathrm{d}s}{\sqrt{x^2+y^2}}$$

where C is the unit square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$.



Solution
We can break the integral into four parts

$$\oint_C = \int_A^B + \int_B^D + \int_D^E + \int_E^A$$

- Along AB $y = 1$ and $\mathrm{d}s = -\mathrm{d}x$
- Along BD $x = -1$ and $\mathrm{d}s = -\mathrm{d}y$
- Along DE $y = -1$ and $\mathrm{d}s = \mathrm{d}x$
- Along EA $x = 1$ and $\mathrm{d}s = \mathrm{d}y$

Thus the integral becomes

$$\begin{aligned}\oint_C \frac{\mathrm{d}s}{\sqrt{x^2+y^2}} &= \int_1^{-1} \frac{-\mathrm{d}x}{\sqrt{1+x^2}} + \int_1^{-1} \frac{-\mathrm{d}y}{\sqrt{1+y^2}} + \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1+x^2}} + \int_{-1}^{1} \frac{\mathrm{d}y}{\sqrt{1+y^2}} \\ &= 4\int_{-1}^{1} \frac{\mathrm{d}t}{\sqrt{1+t^2}} = 4[\sinh^{-1} t]_{-1}^{1} = 8\sinh^{-1} 1\end{aligned}$$

4.3.2 Surface integrals

We recall the definition of an integral of a function $f(x)$ from Engineering Analysis 1,

$$\int_a^b f(x)\mathrm{d}x = \lim_{\substack{n\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i$$

where $a = x_0 < x_1 < \cdots < x_n = b$, $\Delta x_i = x_i - x_{i-1}$ and $x_{i-1} \le \bar{x}_i \le x_i$. We remember that this integrals is equal to the area under the curve $f(x)$ between $x = a$ and $x = b$, as shown in Figure 4.3.

We now wish to extend this to integrals of functions of more than one variable. Next we consider $z = f(x, y)$ and a region R of the xy plane, as illustrated in Figure 4.4. We define the

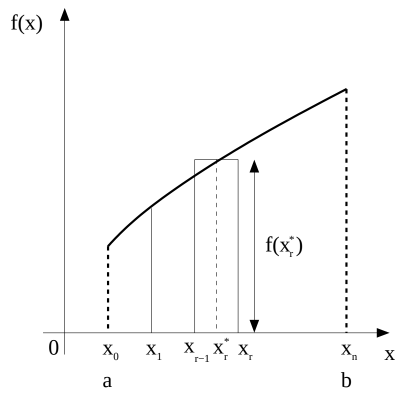


Figure 4.3: Integral of a function of a single variable

integral of $f(x, y)$ over R by

$$\int\int_R f(x,y)\mathrm{d}A = \lim_{\substack{n \to \infty \\ \Delta A_i \to 0}} \sum_{i=1}^{n} f(\bar{x}_i, \bar{y}_i)\Delta A_i \tag{4.19}$$

where ΔA_i is an elemental area of R and $(\bar{x}_i, \bar{y}_i)$ is a point in ΔA_i. As we have already seen $f(x, y)$ represents a surface and so $f(\bar{x}_i, \bar{y}_i)\Delta A_i = \bar{z}_i\Delta A_i$ is the volume between the $z = 0$ and $z = \bar{z}_i$ whose base cross section is ΔA_i. The integral is the limit of the sum of all such volumes and so it is the volume under the surface of $z = f(x, y)$ above R.

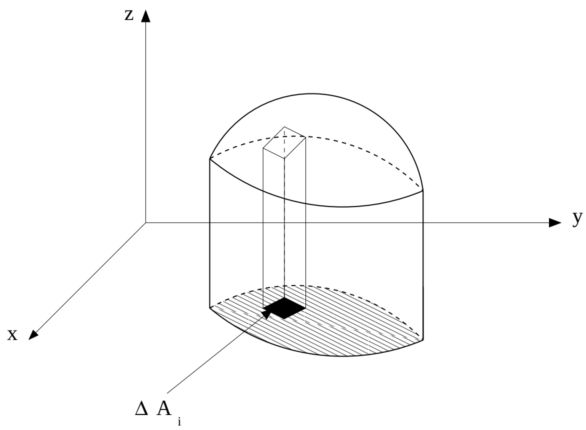


Figure 4.4: Integral of a function of two variables

If we introduce a series of lines which are parallel to the x and y axis, as shown on Figure 4.5, we can write $\Delta A_i = \Delta x_i \Delta y_i$, giving

$$\int\int_R f(x,y)\mathrm{d}A = \int\int_R f(x,y)\mathrm{d}x\mathrm{d}y = \lim_{n\to\infty} \sum_{i=1}^{n} f(\bar{x}_i, \bar{y}_i)\Delta x_i \Delta y_i \tag{4.20}$$

Note that we can evaluate integrals of the type $\int\int_R f(x, y)\mathrm{d}x\mathrm{d}y$ as repeated single integrals in x and y and consequently they are usually called **double integrals**. For the particular case of the integral $\int\int_R \mathrm{d}A$ we note that this equal to the area of region R.

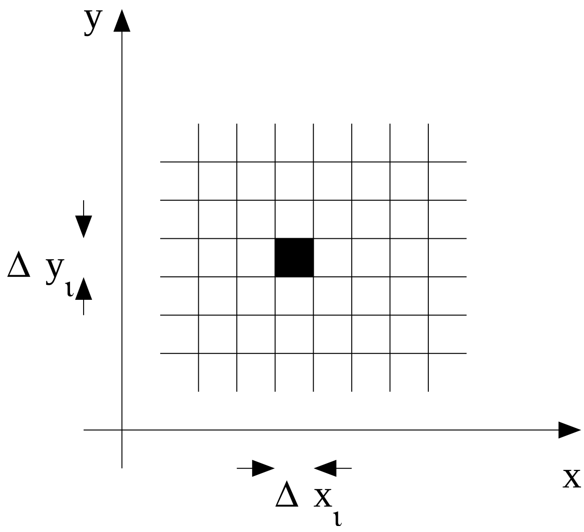


Figure 4.5: Lines introduced for double integrals

There are two alternatives to evaluating double integrals. If data is given such that $y = g(x)$, ie y is some function of x then we work out the integral by first performing the integration with respect to y and then with respect to x, ie

$$\int\int_R f(x,y)\mathrm{d}A = \int_a^b \left[\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\mathrm{d}y\right]\mathrm{d}x$$

Alternatively if we have that x is expressed as some function of y, eg $x = h(y)$, then we first perform the integration with respect to x and then integrate with respect to y

$$\int\int_R f(x,y)\mathrm{d}A = \int_c^d \left[\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\mathrm{d}x\right]\mathrm{d}y \tag{4.21}$$

In the particular case where the region R is a rectangle, then the limits of the integration are constant and so it does not matter whether integrate x or y first.

$$\int\int_R f(x,y)\mathrm{d}A = \int_c^d\int_a^b f(x,y)\mathrm{d}x\mathrm{d}y = \int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x$$

Example
Evaluate the integral

$$\int_0^1 \int_1^3 (x^2 + y^2)\mathrm{d}x\mathrm{d}y$$

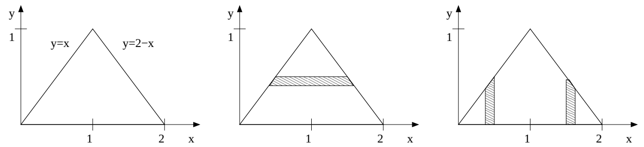
Solution
If we integrate with respect to x first, then we obtain

$$\begin{aligned}\int_0^1 \int_1^3 (x^2 + y^2)\mathrm{d}x\mathrm{d}y &= \int_0^1 \left[\frac{1}{3}x^3 + y^2 x\right]_{x=1}^{x=3} \mathrm{d}y \\ &= \int_0^1 \left(\frac{26}{3} + 2y^2\right) \mathrm{d}y = \left[\frac{26}{3}y + \frac{2}{3}y^3\right]_0^1 = \frac{28}{3}\end{aligned}$$

Alternatively with respect to y first

$$\begin{aligned}\int_0^1 \int_1^3 (x^2 + y^2)\mathrm{d}x\mathrm{d}y &= \int_1^3 \left[x^2 y + \frac{1}{3}y^3\right]_{y=0}^{y=1} \mathrm{d}x \\ &= \int_1^3 \left(x^2 + \frac{1}{3}\right) \mathrm{d}x = \frac{28}{3}\end{aligned}$$

Example
Evaluate $\int\int_R (x^2 + y^2)\mathrm{d}A$ over a triangle with vertices $(0, 0)$, $(2, 0)$ and $(1, 1)$.
Solution



First, integrating with respect to x first gives

$$\begin{aligned}\int\int_R (x^2 + y^2)\mathrm{d}A &= \int_0^1 \int_{x=y}^{x=2-y} (x^2 + y^2)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \left[\frac{1}{3}x^3 + y^2 x\right]_{x=y}^{x=2-y} \mathrm{d}y = \int_0^1 \left(\frac{8}{3} - 4y + 4y^2 - \frac{8}{3}y^3\right) \mathrm{d}y = \frac{4}{3}\end{aligned}$$

Next integrating with respect to y first

$$\int\int_R (x^2 + y^2)\mathrm{d}A = \int_0^1 \int_{y=0}^{y=x} (x^2 + y^2)\mathrm{d}y\mathrm{d}x + \int_1^2 \int_{y=0}^{y=2-x} (x^2 + y^2)\mathrm{d}y\mathrm{d}x$$

Here the integrals are

$$\begin{aligned}\int_0^1 \int_{y=0}^{y=x} (x^2 + y^2)\mathrm{d}y\mathrm{d}x &= \int_0^1 \left[x^2 y + \frac{1}{3}y^3\right]_{y=0}^{y=x} \mathrm{d}x = \int_0^1 \frac{4}{3}x^3 \mathrm{d}x = \frac{1}{3} \\ \int_1^2 \int_{y=0}^{y=2-x} (x^2 + y^2)\mathrm{d}y\mathrm{d}x &= \int_1^2 \left[x^2 y + \frac{1}{3}y^3\right]_{y=0}^{y=2-x} \mathrm{d}x = \int_1^2 \left(\frac{8}{3} - 4x + 4x^2 - \frac{4}{3}x^3\right) \mathrm{d}x = 1\end{aligned}$$

So $\int\int_R (x^2 + y^2)\mathrm{d}A = 1 + \frac{1}{3} = \frac{4}{3}$.

4.3.3 Volume integrals

Volume integrals are evaluated by carrying out three successive integrals. Volume integrals are of the form

$$\int\int\int_V \mathrm{d}V \tag{4.22}$$

and are called **triple integrals**. They are evaluated in the same way as double integrals, we start by evaluating the inner integral and work outwards. The main difficulty is associated with determining the correct limits for the integration. To aid, this one may make a sketch of the region to be integrated. Also useful to note that if integrals are evaluated in the order x, y, z then the limits on the y integral may depend on z but not on x.

Example

A cube $0 \leq x, y, z \leq 1$ has a variable density given by $\rho = 1 + x + y + z$, what is the total mass of the cube

Solution

The total mass is given by

$$\begin{aligned}
M &= \int\int\int_V \rho \mathrm{d}V \\
&= \int_0^1\int_0^1\int_0^1 (1+x+y+z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1\int_0^1 \left[x + \frac{x^2}{2} + xy + xz\right]_0^1 \mathrm{d}y\mathrm{d}z \\
&= \int_0^1\int_0^1 \left(\frac{3}{2} + y + z\right)\mathrm{d}y\mathrm{d}z = \int_0^1 \left[\frac{3y}{2} + \frac{y^2}{2} + yz\right]_0^1 \mathrm{d}z \\
&= \int_0^1 (2+z)\mathrm{d}z = \left[2z + \frac{z^2}{2}\right]_0^1 = \frac{5}{2}
\end{aligned}$$

4.4 Summary and Further Reading

The key topics from this chapter and references to further reading in James (fourth edition) and Croft and Davison (second edition) are

- Partial differentiation. Croft and Davison [pg 990-1001]. James [pg 715-733]
- Total differentiation. James [pg 733-739].
- Integration of lines, surfaces and volumes. Croft and Davison [pg 863-868]. James [pg 646-657].

Chapter 5

Sequences and Series

This chapter investigates sequences and series and there importance in engineering. Sequences are important and arise if a continuous function is measured or sampled at periodic intervals. They also arise when attempts are made to find approximate solutions of equations that model physical phenomena. Closely related to sequences are series. They are important as certain mathematical problems can be expressed as series. Two well known series that we shall consider are the Taylor and Maclaurin series.

5.1 Sequences and Series

A **sequence** is a set of numbers which are written down in a specific order. Examples of sequences are $2, 4, 6, 8$ and $-7, -9, -11, -13$. We call each number a **term** of the sequence. The continuation dots $\cdots$ are sometimes used to illustrate that the sequence continues.

Often sequences arise from the evaluation of a function, for example if we consider the set of whole numbers $\{0, 1, 2, 3, \cdots\}$, the set of values $\{f(0), f(1), f(2), f(3), \cdots\}$ which arise from evaluating the function on the set of whole numbers is also called a sequence. In this case, we give the identify terms in the sequence as follows $f_0 = f(0)$, $f_1 = f(1)$ and so on. Thus the first term in the sequence is f_0, the second term in the sequence is f_1. If the sequence has a given number of terms such as $\{f_0, f_1, \cdots, f_n\}$ we call it a **finite** sequence. Sequences like $\{f_0, f_1, \cdots, f_\infty\}$ which extend to infinity are called **infinite** sequences.

If the next term in a sequence can be generated from some combination of previously computed terms, the formula which gives the next term is called a **recurrence relation**.

Example
One way to compute square roots is Newton formula. This states that if x is an approximation to the square root of a, then a/x is also an approximation to $\sqrt{a}$. A better approximation can be obtained by taking an average of the two values. Thus if x_0 is an approximation to $\sqrt{a}$ then

$$x_1 = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right)$$

similarly

$$x_2 = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right)$$

is a better approximation than x_0. In general x_{n+1} given by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

is better approximation than x_n, this is an example of recurrence relation. If we wish to compute $\sqrt{2}$ then starting with $x_0 = 1$ gives the sequence

$$x_0 = 1 \qquad x_1 = \frac{3}{2} = 1.5 \qquad x_2 = \frac{17}{12} = 1.416666(6dp) \qquad x_3 = \frac{577}{408} = 1.414216(6dp)$$

A **series** is obtained when terms of a sequence are added. For example, if a sequence contains $2, 4, 6, 8, 10$, then by adding the terms we obtain the series

$$2 + 4 + 6 + 8 + 10$$

We can use sigma notation to write a series more concisely. For example, if a sequence contains the integers $0, 1, 2, \cdots, n$ a series is given by

$$S_n = 0 + 1 + 2 + \cdots + n = \sum_{k=0}^{n} k$$

Example
Use summation notation to write the series consisting of a) the first six odd numbers and b) the first seven even numbers.
Solution
a) A series which sums the first six odd numbers is given by

$$\sum_{k=1}^{6}(2k-1) = 1 + 3 + 5 + 7 + 9 + 11$$

b) A series which sums the first seven even numbers is given by

$$\sum_{k=1}^{7} 2k = 2 + 4 + 6 + 8 + 10 + 12 + 14$$

5.1.1 Graphical representation of sequences

Sometimes it helpful to display a sequence graphically. We can do this by plotting each term in the sequence on a standard x, y graph. For example, terms in a particular sequence are defined by $x_n = 1 + (-1)^n/n$, starting with $n = 1$ and considering terms up to $n = 10$ gives to $2dp$

$$0,\ 1.50,\ 0.67,\ 1.25,\ 0.80,\ 1.17,\ 0.86,\ 1.12,\ 0.89,\ 1.10$$

By plotting each term of the sequence as a graph, where the term's index is used as the x coordinate and the term's value is used as the y coordinate, gives the plot shown in Figure 5.1. From

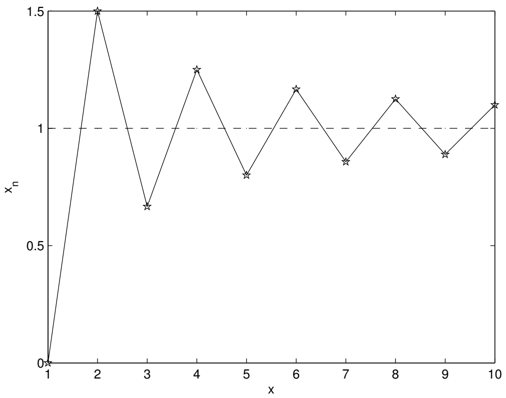


Figure 5.1: Graph of the sequence $x_n = 1 + (-1)^n/n$

this figure, we can observe that values of the sequence oscillate around 1 and become closer to 1 as n increases. Thus plotting a sequence can often give us valuable insights in to its behaviour.

5.2 Finite sequences and series

We now wish to look at finite sequences and series in more detail.

5.2.1 Arithmetical sequences and series

An **arithmetical sequence** is a sequence in which the difference between successive terms is a constant number. Examples of arithmetical sequences are $\{0, 3, 6, 9, 12, 15\}$ and $\{1, 0, -1, -2, -3\}$. Traditionally arithmetical series were called **arithmetical progressions**, however the former name is now preferred. We can write arithmetical sequences as $\{a + kd\}_{k=0}^{n-1}$ where a is the first term, d is the difference between the terms and n is the number of terms in the sequence. So, for the first example $a = 0$, $d = 3$ and $n = 6$, for the second example $a = 1$, $d = -1$ and $n = 5$.

The sum of terms in an arithmetical sequence is an **arithmetical series**. In general this can be written as

$$S_n = a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \sum_{k=0}^{n-1}(a + kd) \tag{5.1}$$

We can obtain an expression for the sum of n terms in the series. If we expand the summation and then write it in reverse order we have

$$\begin{array}{ccccccccccc} S_n & = & a & + & (a+d) & + & (a+2d) & + & \cdot & + & [a+(n-1)d] \\ S_n & = & [a+(n-1)d] & + & [a+(n-2)d] & + & [a+(n-3)d] & + & \cdots & & a \end{array}$$

Now if we add these expressions we obtain

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] + \cdots [2a + (n - 1)d]$$

Thus giving

$$S_n = \frac{1}{2}n[2a + (n-1)d] \tag{5.2}$$

as the sum of n terms of an arithmetical series.

Example
How many terms of the arithmetical sequence $2, 4, 6, 8, \cdots$ will give rise to 420?
Solution
For this example, $a = 2$, $d = 2$, $S_n = 420$, we need to find n

$$S_n = 420 = \frac{1}{2}n[4 + 2(n-1)] = 2n + n(n-1)$$

Thus

$$n^2 + n - 420 = 0 \qquad n = \frac{-1 \pm \sqrt{1 - 4(-420)}}{2}$$

Hence $n = 20$ or $n = -22$, since n must be a positive number, $n = 20$

Example
A building company offers to place a foundation pile at a cost of 100 pounds for the first metre, 110 pounds for the second metre and increasing at a cost of 10 pounds per metre thereafter. It is decided to set piles at 5 metres.

a) What is the the total cost of the piling?

b) What is the cost of piling the last metre?

Solution
a) The cost is the sum of the arithmetic series where $a = 100$, $b = 10$ and $n = 5$

$$S_n = \frac{5}{2}[2(100) + (5-1)10] = 600 \text{ pounds}$$

b) The cost of piling the last metre is given by the fifth term in the sequence. This $100 + (5 - 1)10 = 140$ pounds.

5.2.2 Geometric sequences and series

A **geometric sequence** is one in which the ratio of successive terms is a constant number. Examples of geometric sequences are $\{3, 6, 12, 24, 48\}$ and $\{-1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}, -\frac{1}{32}\}$. A geometric sequence always takes the form $\{ar^k\}_{k=0}^{n-1}$ where a is the first term in the sequence, r is the ratio between the terms and n is the number of terms in the sequence. Thus in the first example $a = 3$, $r = 2$ and $n = 5$, for the second example $a = -1$, $r = \frac{1}{2}$ and $n = 5$. Geometric sequences are sometimes still called **geometric progressions**. The sum of a geometric sequence is a **geometric series**. The general geometric series has the form

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k \tag{5.3}$$

To obtain the sum S_n, we first multiply the equation by r

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$

then if we subtract this from S_n we obtain

$$(1-r)S_n = a - ar^n$$

so that

$$S_n = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r} \tag{5.4}$$

Example
An insurance company guarantees that, for a fixed annual premium payable at the beginning of each year, for a period of 25 years, the return will be equal to the premium paid together with 3% compound interest. For an annual premium of 250 pounds what is the guaranteed sum at the end of 25 years?
Solution
The first year premium earns interest for 25 years and so grantees $250(1+0.03)^{25}$
The second year premium earns interest for 24 years and so grantees $250(1+0.03)^{24}$
$\vdots$
The final year premium earns interest for 1 year and so grantees $250(1+0.03)$
The total sum is therefore

$$250[(1.03)+(1.03)^2+\cdots+(1.03)^{25}]$$

the term inside the square brackets is a geometric sequence. Taking $a=1.03$, $r=1.03$ and $n=25$ gives the total cost as

$$250\left[1.03\frac{(1-1.03^{25})}{(1-1.03)}\right] = 9388 \text{ pounds}$$

5.2.3 Other finite series

Sometimes engineers are required to use finite series other than arithmetical and geometrical sequences. We investigate a method that can be generalised to finding the sums of different finite series and apply it to the case of finding the sum of squares.

We wish to find the summation of squares

$$S_n = 1^2+2^2+3^2+\cdots+n^2 = \sum_{k=1}^{n} k^2$$

To do this we use the identity $(k+1)^3-k^3 = 3k^2+3k+1$. This means we can write

$$\sum_{k=1}^{n}[(k+1)^3-k^3] = \sum_{k=1}^{n}(3k^2+3k+1)$$

In this expression, we can expand the left hand side to find that

$$2^3-1^3+3^3-2^3+4^3-3^3+\cdots+(n+1)^3-n^3 = (n+1)^3-1$$

and the right hand side is equal to

$$3\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

We already know that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ and $\sum_{k=1}^{n} 1 = n$ so this means that

$$(n+1)^3-1 = 3\sum_{k=1}^{n}k^2+\frac{3n}{2}(n+1)+n$$

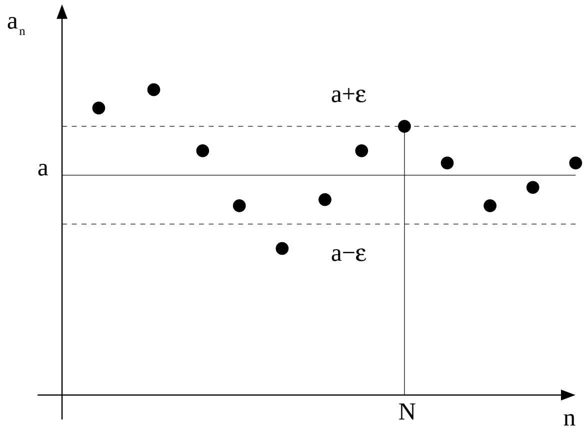


Figure 5.2: Convergence of a sequence

which finally gives

$$S_n = \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) \tag{5.5}$$

5.3 Limit of a sequence

5.3.1 Convergent sequences

We previously saw how we could use Newton's formula to gain an ever improving approximation to the square root of a value. Starting with 1 we obtained the following improving approximations to $\sqrt{2}$

$$x_0 = 1$$
$$x_1 = 1.50$$
$$x_2 = 1.42$$

if the process is continued we would obtain

$$x_{22} = 1.41$$
$$x_{23} = 1.41$$

indeed for $n \geq 22$ we have $x_n = 1.41$ to 2dp. We observe that the difference between x_{22} and x_{23} is indistinguishable when the numbers are expressed to two decimal places, in other words the difference is less than the rounding error. When this happens, we say that the sequence **tends to a limit** or **has a limiting value** or **converges** or that it **is convergent**.

Given a general sequence $\{a_n\}_{n=0}^{\infty}$ we say it has the limiting value a as n becomes large, if given a small positive number ϵ, a_n differs from a by less than ϵ for all sufficiently large n, ie

$a_n \to a$ as $n \to \infty$ if, given any $\epsilon > 0$, there is a number N such that $|a - a_n| < \epsilon$ for all $n > N$

We remark that $\to$ stands for 'tend to the value' or 'converges to the limit'. An alternative notation would be to write

$$\lim_{n\to\infty} a_n = a \tag{5.6}$$

We illustrate this process graphically in Figure 5.2.

Note that the limit of a sequence need not actually be an element of the sequence. For example $\{n^{-1}\}_{n=1}^{\infty}$ has the limit of 0, but 0 is not an element of the sequence.

5.3.2 Proprieties of convergent sequence

It turns out that a convergent sequence satisfies a number of properties which are given below

- Every convergent sequence is bounded; that is, if $\{a_n\}_{n=0}^{\infty}$ is convergent then there is a positive number M such that $|a_n| < M$ for all n.
- If $\{a_n\}$ has limit a and $\{b_n\}$ has limit b then
  - $\{a_n + b_n\}$ has limit $a + b$
  - $\{a_n - b_n\}$ has limit $a - b$
  - $\{a_n b_n\}$ has limit ab
  - $\{a_n / b_n\}$ has limit a/b for $b_n \neq 0$ and $b \neq 0$.

Example
Find the limits of the sequence $\{x_n\}_{n=0}^{\infty}$ when x_n is given by

$$\text{a) } x_n = \frac{n}{n+1} \qquad \text{b) } x_n = \frac{2n^2 + 3n + 1}{5n^2 + 6n + 2}$$

Solution
a) $x_n = n/(n+1)$ leads to the sequence $\{0, \frac{1}{2}, \frac{3}{4}, \frac{4}{5}, \cdots\}$. Already from these values it seems that $x_n \to 1$ as $n \to \infty$. We can prove this by rewriting x_n as

$$x_n = 1 - \frac{1}{n+1}$$

As n increases $1/(n+1)$ becomes smaller and smaller, thus we have

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

b) Now considering $x_n = \frac{2n^2+3n+1}{5n^2+6n+2}$ it is easiest to divide the numerator and denominator by the highest power of n, giving

$$x_n = \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{5 + \frac{6}{n} + \frac{2}{n^2}}$$

We have that $\lim_{n\to\infty} 2 + \frac{3}{n} + \frac{1}{n^2} = 2$ and $\lim_{n\to\infty} 5 + \frac{6}{n} + \frac{2}{n^2} = 5$. Hence we have that

$$\lim_{n\to\infty} \frac{2n^2 + 3n + 1}{5n^2 + 6n + 2} = \frac{2}{5}$$

5.3.3 Divergent sequences

To illustrate the fact that not all sequences converge we consider the following geometric sequence

$$a_n = r^n \qquad r \text{ constant} \tag{5.7}$$

For this sequence we have

$$\lim_{n\to\infty} a_n = \begin{cases} 0 & (-1 < r < 1) \\ 1 & (r = 1) \end{cases}$$

if $r > 1$ the sequence increases without bound as $n \to \infty$ and we say that it **diverges**. If $r = -1$ the sequence takes the values of alternating ± 1, and has no limiting value. If $r < -1$ the sequence is unbounded and the terms alternate in sign.

5.3.4 Cauchy's test for convergence

The following test for convergence is used in a computational context. If we do not know the limit a to which a sequence $\{a_n\}$ converges we cannot measure $|a - a_n|$. However, in a computational context where we often use a recurrence relationship to compute the sequence $\{a_n\}$, we say that is has converged when all subsequent terms yield the same level of approximation required. We say that a sequence of finite terms is convergent if for any n and $m > N$

$$|a_n - a_m| < \epsilon \tag{5.8}$$

where ϵ is specified. This means that a sequence turns to a limit if all the terms of the sequence for $n > N$ are restricted to an interval that can be made arbitrarily small by making N arbitrarily large. This is called **Cauchy's test for convergence**.

5.4 Infinite Series

We must exercise care when dealing with infinite series as mistakes can be made if they are not dealt with correctly. If we consider the series

$$S = 1 - 2 + 4 - 8 + 16 - 32 + \cdots$$

then by multiplying it by 2 we obtain

$$2S = 2 - 4 + 8 - 16 + 32 - 64 + \cdots$$

if we add these equations we might come to the conclusion that $3S = 1$ or $S = \frac{1}{3}$, however this result is clearly incorrect. To avoid making such mistakes we have introduce methods for dealing with infinite series correctly.

5.4.1 Convergence of an infinite series

We have already seen that series and sequences are closely related. When the sum S_n of a series of n terms tends to a limit as $n \to \infty$ we say it is **convergent**. Provided that we can express S_n is a simple form it is usually easy to say whether or not the series converges. When considering infinite series, the sequence of partial terms is taken to the limit.

Example
We wish to examine the following series for convergence

$$
\begin{array}{ll}
a) & 1+3+5+7+9+\cdots \\
b) & 1^2+2^2+3^2+4^2+ \\
c) & 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots \\
d) & \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots
\end{array}
$$

a) The first case is an arithmetic sequence which we can write as

$$S_n = \sum_{k=0}^{n-1}(2k+1) = 1+3+5+\cdots+(2n-1) = n^2$$

we can see that $S_n \to \infty$ as $n \to \infty$ and the series does not converge to a limits. It is an example of a divergent series.
b) The second case can be written as

$$S_n = 1^2+2^2+3^2+\cdots+\cdots n^2 = \frac{1}{6}n(n+1)(2n+1)$$

This is another example where $S_n \to \infty$ as $n \to \infty$, ie the series is divergent.
c) For the third example

$$S_n = 1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}$$

we have a geometric sequence, the sum can be written as

$$S_n = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right)$$

we have that as $n \to \infty$, $S_n \to 2$, hence the sum converges to 2.
d) In the final example we have

$$S_n = \frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots+\frac{1}{n(n+1)} = \sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k+1}$$

Expanding we have

$$S_n = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{n}-\frac{1}{n+1} = 1-\frac{1}{n+1}$$

thus as $n \to \infty$, $S_n \to 1$, hence the sum converges to 1.

5.4.2 Tests of convergence of positive series

Unfortunately the sum of a series can't always be expressed in a closed form expression. For such cases we use a series of tests to examine the convergence of a series.

Comparison Test

Given a sequence $\sum_{k=0}^{\infty} c_k$ which consists of positive terms ($c_k \geq 0$ for all k) which is convergent, then if we have a different series, $\sum_{k=0}^{\infty} u_k$ of positive terms such that $u_k \leq c_k$ then $\sum_{k=0}^{\infty} u_k$ is convergent also. Note that if $\sum_{k=0}^{\infty} c_k$ diverges and $u_k \geq c_k \geq 0$ for all k then $\sum_{k=0}^{\infty} u_k$ also diverges.

D'Alembert's ratio test

Given a series of positive terms $\sum_{k=0}^{\infty} u_k$ and that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \ell$ exists. Then D'Alembert's ratio test says that this series is convergent if $\ell < 1$ and divergent if $\ell > 1$. For $\ell = 1$ it is not possible to use D'Alembert test to determine whether the series converges or not.

Example
Determine whether the following sequences are convergent

$$\text{a)} \sum_{k=0}^{\infty} \frac{2^k}{k!} \qquad \text{b)} \sum_{k=0}^{\infty} \frac{2^k}{(k+1)^2}$$

Solution
a) Using D'Alembert's ratio test we write $u_n = 2^n/n!$ giving

$$\frac{u_{n+1}}{u_n} = \frac{2^{k+1}n!}{2^k(n+1)!} = \frac{2}{n+1}$$

which tends to zero as $n \to \infty$. Thus the series is convergent.
b) Again using D'Alembert's ratio test we have

$$\lim_{n\to\infty} \left[\frac{2^{n+1}(n+1)^2}{2^n(n+2)^2}\right] = \lim_{n\to\infty} \left[2 - \frac{4}{n+2} + \frac{2}{(n+2)^2}\right] = 2$$

which follows by using partial fractions, hence the series diverges.

Necessary condition for convergence

This states that for convergence of any series we need that the terms of the series must tend to zero as $n \to \infty$. One way to test for divergence is that if $u_n \to u \neq 0$ as $n \to \infty$ then $\sum_{k=0}^{\infty} u_k$ is divergent.

Care is required, since although $u_n \to 0$ as $n \to \infty$ is required for convergence it does not guarantee convergence!

5.4.3 Absolute convergence of a general series

We have just spoken about a number of tests can be applied when our series has positive terms. In general a series S given by

$$S = \sum_{k=0}^{\infty} u_k$$

may have both positive and negative terms. Now, if we consider the series

$$T = \sum_{k=0}^{\infty} |u_k|$$

and it turns out that T is convergent, then we say that S is **absolutely convergent**.

We can test for absolute convergence by extending our earlier convergence tests for positive series. For example, extending the D'Alembert test we have

$$\text{if} \quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \quad \text{then} \ \sum_{k=0}^{\infty} u_k \text{ is absolutely convergent}$$

$$\text{if} \quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1 \quad \text{then} \ \sum_{k=0}^{\infty} u_k \text{ is divergent}$$

$$\text{if} \quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1 \quad \text{then no conclusion can be made}$$

The product of two absolutely convergent series $A = \sum a_n$ and $B = \sum b_n$ is also an absolutely convergent series.

There are convergent sequences that are not absolutely convergent series, the most common series of this type are alternating series, where u_n changes in sign. For example if

$$|u_n| < |u_{n-1}| \qquad \text{for all } n \text{ and } u_n \to 0 \text{ as } n \to \infty$$

then the series is convergent even if the series is not absolutely convergent.

5.5 Power Series

A power series is a series of the type

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

where $a_0, a_1, a_2, \cdots$ are independent of x

5.5.1 Convergence of power series

Power series often converge for certain value for x, and diverge for others. We can use D'Alembert's ratio test to investigate the convergence of power series

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| < 1$$

Thus the sequences converges if

$$|x| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \qquad \text{or} \qquad |x| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Another way to interpret this is if we denote $r = \lim_{n\to\infty} |a_n/a_{n+1}|$ then we see that the series is absolutely convergent for $-r < x < r$ and divergent outside these limits, ie $x > r$ and $x < -r$. The convergence behaviour at $x = \pm r$ has to be determined by other means. We call r the **radius of convergence**.

5.5.2 Binomial Series

The first power series that we wish to explore in more detail is the binomial series. If $n > 0$ is some positive integer then we can write expansions of $(1 + x)^n$ as

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

this says that we can expand $(1+x)^n$ into n terms. We call this the **Binomial theorem** or **Binomial Series**. When n is no longer a positive integer, but is some real number, we get a alternative form of the Binomial form which consists of an infinite series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \qquad \textbf{only for } -1 < x < 1$$

Note that when n is *any* real number the series is infinite and only valid for $-1 < x < 1$.

Example
Obtain the form of the expansion of $1/(\ell - x)^2$ by writing down the first four terms. Write down the condition required for convergence of the series.
Solution
First we write

$$\frac{1}{(\ell-x)^2} = \frac{1}{\ell^2\left(1-\frac{x}{\ell}\right)^2} = \frac{1}{\ell^2}\left(1-\frac{x}{\ell}\right)^{-2}$$

Now we can use the Binomial theorem to expand $\left(1-\frac{x}{\ell}\right)^2$

$$\begin{aligned}\left(1-\frac{x}{\ell}\right)^{-2} &= 1 + (-2)\left(-\frac{x}{\ell}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{x}{\ell}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{x}{\ell}\right)^3 + \cdots \\ &= 1 + \frac{2x}{\ell} + \frac{3x^2}{\ell^2} + \frac{4x^3}{\ell^3} + \cdots\end{aligned}$$

Thus

$$\frac{1}{\ell^2}\left(1-\frac{x}{\ell}\right)^{-2} = \frac{1}{\ell^2} + \frac{2x}{\ell^3} + \frac{3x^2}{\ell^4} + \frac{4x^3}{\ell^5} + \cdots$$

Therefore $a_n = n/\ell^{n+1}$, so

$$|x| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n\ell^{n+2}}{(n+1)\ell^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n\ell}{(n+1)}\right| = \lim_{n\to\infty}\left|\ell - \frac{\ell}{(n+1)}\right| = |\ell|$$

Hence $|x| < |\ell|$ for convergence.

5.5.3 Maclaurin Series

The next series that we wish to consider is the **Maclaurin series**. This takes the form

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \tag{5.9}$$

which is an infinite series, although good approximations can often be obtained by using just a few terms.

Example
Write down the first four terms of the Maclaurin series for $f(x) = e^x$
Solution
Using equation (5.9) we have

$$\begin{aligned}e^x &= e^0 + xe^0 + \frac{x^2}{2!}e^0 + \frac{x^3}{3!}e^0 + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\end{aligned}$$

We can apply the Maclaurin series to obtain a range series for common functions. Some examples are given below

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \\
e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^n \frac{x^{n+1}}{n+1} + \cdots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots
\end{aligned}$$

which are valid for all x. Note that for $\cos x$ and $\sin x$, the expansions are only valid when x is measured in radians.

Small angle approximation

An important consequence of these series expansions is the small angle approximation of the $\cos$ and $\sin$ functions. If x is small, and measured in radians, then we can approximate $\cos x$ and $\sin x$ by

$$\cos x \approx 1 - \frac{x^2}{2} \qquad \sin x \approx x \tag{5.10}$$

we call this the **small angle approximation**. These approximations make sense since if x is sufficiently small. Higher order powers of x such as x^3, x^4, ... quickly become very small.

5.5.4 Taylor Series

The Taylor series is very similar to the Maclaurin series. Instead of expanding $f(x)$ about the origin, we now expand it about some point $x = a$, giving the **Taylor series**

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \tag{5.11}$$

Note that if we substitute $a = 0$ into this equation we get the Maclaurin series. The series is clearly an infinite series. Taylor series are commonly written also in the following form

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x) \tag{5.12}$$

where R_n is called the remainder and is defined as follows

$$R_n = \frac{(x-a)^{n+1}}{(n+1)!}f^{n+1}(\xi) \qquad a < \xi < x \tag{5.13}$$

which says that all the remaining terms in the infinite series can be summed to form the remainder. The remainder consists of working out the $n+1$th derivative of $f(x)$ at some *unknown* point lying between a and x.

Taylor series is often used in the derivation of approximate numerical methods, we consider a simple application in the following example.

Example
Consider two points $x = a$ and $x = a + h$. Use Taylor series to find a simple way to approximate the derivative to the function $f(x)$ at the point $x = a$.
Solution
The simplest approximation to the derivative will just involve the x coordinates and evaluation of the function. If we write a Taylor series expansion for the point $x = a + h$ we obtain

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots$$

We can rewrite this as

$$f(a+h) = f(a) + hf'(a) + O(h^2)$$

where the $O(h^2)$ stands for the fact that additional terms involve terms like h^2 and higher powers of h. Now if h is sufficiently small we can ignore these higher order terms and write

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

which gives us an approximation to the derivative at $x = a$.

Taylor series for functions of more than one variable

We note that Taylor series can also be extended to functions of more than one variable. For example, Taylor series for functions of two variables is given by

$$\begin{aligned} g(x,y) &= g(a,b) + \frac{1}{1!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right) g(a,b) + \\ &+ \frac{1}{2!}\left((x-a)^2\frac{\partial^2}{\partial x^2} + (x-a)(y-b)\frac{\partial^2}{\partial x \partial y} + (y-b)^2\frac{\partial^2}{\partial y^2}\right) g(a,b) + \cdots \end{aligned}$$

which is often used in deriving approximate numerical methods for problems involving functions of two variables.

5.6 Summary and Further Reading

The key topics from this chapter and references to further reading in James (fourth edition) and Croft and Davison (second edition) are

- Finite sequences and series. Croft and Davison [pg 879-880, 885-887, 890-892]. James [pg 474-481]
- Limits of a sequence. Croft and Davison [pg 881-882]. James [pg 494-501].
- Infinite series. Croft and Davison [pg 881-884]. James [pg 502-509].
- Power series. Croft and Davison [pg 893-906]. James [pg 509-518].