Engineering Analysis 1

Dr. P.D. Ledger

Associate Professor
Civil and Computational Engineering Centre
Swansea University
Singleton Park
Swansea SA2 8PP
Wales. U.K

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Contents

Chapter 1

Numbers Systems

In this chapter we will summarise the concepts and techniques that most students will already understand and extend them in to further developments in mathematics. Indeed, there are three areas where you are already process considerable knowledge

- numbers
- functions
- algebra

These areas are vital to making process in engineering mathematics (indeed they solve many important problems in engineering). We aim to consolidate that knowledge, aim to make it more precise and to develop it.

1.1 Numbers

Mathematics has now developed from primitive arithmetic and geometry and is now a vast body of knowledge. The most ancient mathematical skill is that of counting, first in the form of the natural numbers and later integers. The term **natural numbers** refers to the set $\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\}$ while the term integers to the set $\mathbb{Z} = \{\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$. The integers can be represented as equally spaced points on the **number line** as shown in Figure 1.1. The set of all points (ie not only those represented by integers) represents the **real numbers**, it is denoted by the symbol $\mathbb{R}$. A real number which can be written as the ratio of two integers is called a **rational number** eg $\frac{1}{3}$ and $-\frac{7}{5}$. Other numbers like π and $\sqrt{2}$ which can not be expressed in this way are called **irrational numbers**

Figure 1.1: The number line

In a computer integers can be stored exactly but real numbers can only be stored to a limited number of figures. Computer languages often distinguish between integer values and variables and real values and variables.

The following are known as the **fundamental rules of arithmetic**

- Commutative law of addition $a + b = b + a$
- Commutative law of multiplication $a \times b = b \times a$
- Associative law of addition $(a + b) + c = a + (b + c)$
- Associative law of multiplication $(a \times b) \times c = a \times (b \times c)$
- Distributive law of multiplication over addition and subtraction $(a \pm b) \times c = (a \times c) \pm (b \times c)$
- Distributive law of division over addition and subtraction $(a \pm b) \div c = (a \div c) \pm (b \div c)$

The above operations are called **binary** because they associate with every two members of the set or real numbers a unique and third member.

The devision of two numbers $a \div b$ is frequently written as a/b or $\frac{a}{b}$. When a and b are integers we call $\frac{a}{b}$ a **fraction** and refer to a as the numerator and b as the denominator. Fractions are classified as either proper fractions or improper fractions. When deciding whether a fraction is proper or improper, we ignore negative signs in the numerator or denominator. A proper fraction is when the numerator is less than the denominator, a improper fraction is when the numerator is greater than the denominator. Examples of proper fractions are $\frac{1}{2}$ and $\frac{99}{100}$, examples or improper fractions are $\frac{3}{2}$ and $\frac{100}{89}$. For further information on fractions please refer to the references at the end of this chapter.

The next operation involving real numbers is that of **powering**. For example $a \times a$ is written as a^2 and $a \times a \times a$ as a^3, In general the product of n a's is written as a^n. We call n the **index** or **exponent**. Operations obey the simple rules

- $a^n \times a^m = a^{n+m}$
- $a^n \div a^m = a^{n-m}$
- $(a^n)^m = a^{nm}$

As a consequence of these rules we have $(a^{1/n})^n = a$, $a^0 = 1$ and $a^{1/n} = \sqrt[n]{a}$. In contract to binary operations, the powering operator $(\cdot)^r$ only operates on one element and so is called a **unary** operation.

An order of precedence is observed

- the operation $(\cdot)^r$ is performed first
- then $\times$ or $\div$
- then $+$ or $-$

When two operators of equal precedence are adjacent to each other, the left hand rule is applied. The precedence is over written by brackets.

The number line makes a further property of real numbers, that of ordering, which enables us to make statements like "seven is greater than one" or "five is less than ten". This is made possible by the **comparison symbols** $>$ "greater than" and $<$ "less than". Other comparison symbols that we commonly use are $=$ "equal to", $\neq$ "not equal to", $\geq$ "greater or equal to" and $\leq$ "less than or equal to". Common rules are

- **Rule 1** $(a < b)$ and $(c < d)$ implies $a + c < b + d$
- **Rule 2** $(a < b)$ and $(c > d)$ implies $a - c < b - d$

- **Rule 3** $(a < b)$ and $(b < c)$ implies $a < c$ and $a < b$ implies $a + c < b + c$
- **Rule 4** $(a < b)$ and $(c > 0)$ implies $ac < bc$
- **Rule 5** $(a < b)$ and $(c < 0)$ implies $ac > bc$
- **Rule 6** $(a < b)$ and $(ab > 0)$ implies $\frac{1}{a} > \frac{1}{b}$

Example
Find the value of x for which

$$\frac{1}{2-x} < 1 \tag{1.1}$$

Solution
When $2 - x > 0$, that is when $x < 2$, we can use Rule 4, where we multiply by $2 - x$ to give

$$1 < (2 - x)$$

which reduces to $x < 1$. Which means that (1.1) is satisfied when $x < 2$ and $x < 1$, that is when $x < 1$.
We now consider $2 - x < 0$, that is when $x > 2$. we use Rule 5, where we multiply by $2 - x$ to give

$$1 > (2 - x)$$

so that (1.1) is satisfied when $x > 1$ and $x > 2$, that is when $x > 2$. Thus the inequality is satisfied for values of x such that $x > 2$ or $x < 1$.

The size of a real number x is called its **modulus** and is denoted by $|x|$

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Geometrically speaking $|x|$ is the distance on the number line from the point representing 0, Similarly $|x - a|$ is the distance of the point representing x on the number line from that representing a.

The **set of numbers** between two numbers a and b say, defines the **open interval**. This is the set $\{x : a < x < b, x \text{ in } \mathbb{R}\}$ and is usually denoted by (a, b). Here we use the notation $\{x : P\}$ which means that each value x of the set has the property P. An interval that includes the two end points is called a **closed interval** denoted by $[a, b]$ with

$$[a, b] = \{x : a \leq x \leq b, x \text{ in } \mathbb{R}\}$$

Example
Express the set $\{x : |x + 2| < 5, x \text{ in } \mathbb{R}\}$ as an interval.
Solution
First we note that we can write $|x + 2| < 5$ as $|x - (-2)| < 5$, which means that the distance of the point x on the number line from the point representing -2 is less than 5 units. This implies that $-5 < x + 2 < 5$ which means that $-7 < x < 3$ and the set of numbers which satisfy the inequality is the open interval $(-7, 3)$

We close this section with some miscellaneous results using the modulus

- $|xy| = |x||y|$

- $|x| < a$ with $a > 0$ implies $-a < x < a$
- $|x + y| \leq |x| + |y|$ which is called the Triangular inequality.
- $\frac{1}{2}(x + y) \geq \sqrt{xy}$ when $x \geq 0$ and $y \geq 0$

1.2 Algebra and Geometry

Algebra is made up of well known results such as

- $(a + b)^2 = a^2 + 2ab + b^2$
- $(a + b)(a - b) = a^2 - b^2$

We should already familiar with linear equations written in the form

$$y = mx + c, \tag{1.2}$$

which, when displayed graphically, is a straight line graph where c is the y–intercept and $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ is its gradient. A range of elementary physics can be described by such equations.

We should also be familiar with quadratic equations of the form

$$y = ax^2 + bx + c, \tag{1.3}$$

which has a curved graph where c is the y–intercept, a controls whether the quadratic opens upwards ($a > 0$) or downwards ($a < 0$) as well as the speed of increase of the quadratic and b controls the location of the vertex. Quadratic equations frequently occur in engineering. A common task involves finding the **zeros** or **roots** of a quadratic equation that correspond to the values of x for which y is zero, ie the location(s) at which the graph passes through the x axis.

In order to find a formula for the general roots of a quadratic equation we note the rearrangement

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \tag{1.4}$$

which is known as completing the square.

We can use completing the square to obtain a formula for the general roots of a **quadratic equation** $ax^2 + bx + c = 0$ with $a \neq 0$

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} &= \left(\frac{b}{2a}\right)^2 \qquad \text{by completing the square} \\
\left(x + \frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (1.5)
\end{aligned}$$

This equation gives rise to the following implications

- For $b^2 > 4ac$ we have two real roots
- For $b^2 = 4ac$ we have one repeated root

- For $b^2 < 4ac$ we have no real roots.

Example
Determine the roots of the quadratic equation

$$x^2 + 3x + 2 = 0$$

Solution
To find the roots we simply apply equation (1.5) with $a = 1$, $b = 3$ and $c = 2$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)}}{2} = \frac{-3 \pm 1}{2} = -1 \text{ or } -2$$

1.3 Functions

A function is essentially a way of mathematically expressing the dependence of the value of one quantity, the dependent variable upon another variable, the independent variable. Here quantity can be many different concepts, but in each case we have two different sets X and Y and a rule that assigns to each element x in the set X precisely one element y from the set Y. Whenever this is the case we say that there is a function that maps set X to the set Y. This is written in one of two ways $f : x \to y$ or $y = f(x)$. Engineers prefer the second method while mathematicians the first. In the second case the value or variable win the parenthesis is called the **argument** of the function.

The set X is called the **domain of the function**. The set Y is called the **codomain of the function**. Knowing the size and type of functions in important in computing. In this case it is the type of variables (eg real or integer) and their size that is important. When $y = f(x)$, y is said to be the **image** of x under f. The set of all images $y = f(x)$, x in X is called the **image set** or **range** of f.

The rule giving f is completely determined if we know $f(x)$ and consequently in engineering it is common to refer to the function as $f(x)$ instead of f. Likewise we regard $y = f(x)$ as a variable. However, while x can freely take any value from the set X, the variable $y = f(x)$ depends on the particular element chosen for x. x is therefore called the **independent variable** and y is the **dependent variable**. Graphical representation is obtained by plotting the graph of x against y. It should be obvious that $y = f(x) = ax + b$ and $y = f(x) = ax^2 + bx + c$ are linear and quadratic functions, respectively.

It is worth noting the following points

- In the definition of the function, each input gives rise to exactly one output.
- It is possible for two or more inputs of a function to give rise to the the same output. A function which has the special property that different inputs gives rise to different outputs is said to be **one to one** or **injective**.

- It is possible for one or more elements of the codomain of a function which are not outputs for a function. A function which has that has the special property that every element of its codomain is an output (ie whose range is the whole of its codomain) is said to be **on to** or **surjective**.
- A function which is both injective and surjective is called **bijective**.

Example
The following relationship can be used to convert temperature measured in degrees Celsius (^oC), T_1, in to degrees Fahrenheit (^oF), T_2

$$T_2 = \frac{9}{5}T_1 + 32$$

If we now interpret this as a function with T_1 as the independent variable and T_2 as the dependent variable

a) What are the domain and codomain of the function?

b) What is the function rule?

c) What is image set or range of the function?

d) Use the function to convert the following temperatures in to degrees Fahrenheit $i)$ 60^oC, $ii)$ 0^oC and $iii)$ -50^oC.

Solution
$a)$ Since the temperature can vary continuously, the domain is the set of real numbers such that $T_1 \geq T_0 = -273.16$ (absolute zero). The codomain is the set of real numbers $\mathbb{R}$. Note that if we did not know that T_1 related to temperature in degrees, the domain would simply be the set of all real numbers.
$b)$ The function rule expressed in words is multiply by $\frac{9}{5}$ and then add 32. Algebraically this is $f(T_1) = \frac{9}{5}T_1 + 32$
$c)$ Since the domain is the set $T_1 \geq T_0$, there must be an image for every value of T_1 that is greater than -273.16. Furthermore since every value of T_2 is the image of some value of T_1, it follows that the range of $f(T_1)$ is the set of real numbers greater than -459.688.
$d)$ To perform conversion we use the rule $T_2 = \frac{9}{5}T_1 + 32$ giving $i)$ 140^oF, $ii)$ 32^oF and $iii)$ -58^oF.

It is important to appreciate the difference between a function and a formula. A function is a mapping that associates a unique member of the codomain with every member of its domain. It may also be possible to express this association with a formula, however, some functions may be expressed by different formulae in different parts of their domain.

1.3.1 Inverse functions

Sometimes we need to express the function in the reverse sense. For example given the function

$$T_2 = f(T_1) = \frac{9}{5}T_1 + 32 \tag{1.6}$$

where T_2 is the temperature in Fahrenheit and T_1 is the temperature in degrees Celsius. Rearranging gives

$$T_1 = g(T_2) = \frac{5}{9}(T_2 - 32) \tag{1.7}$$

which reverses the operation performed by the function $T_2 = f(T_1)$ and for this reason it is called the inverse function of $f(T_1)$. Strictly speaking, this formula is valid for all values of $T_2 \geq -459.688$. In general the inverse of a function f is a function that reverses the operation carried out by f. It is denoted by f^{-1}. So if $y = f(x)$ we have that $x = f^{-1}(y)$ where $y = f(x)$. In the above example where $T_2 = f(T_1)$ then $T_1 = f^{-1}(T_2)$.

In the definition of the function $y = f(x)$ we called x the independent variable and y the dependent variable. However, if we consider the inverse function, the independent variable x for f acts as the dependent variable for f^{-1} and correspondingly the dependent variable for f becomes the independent variable for f^{-1}.

Example

Obtain the inverse function of $y = f(x) = \frac{1}{5}(4x - 3)$

Solution

We rearrange $y = f(x) = \frac{1}{5}(4x - 3)$ to make the subject

$$x = f^{-1}(y) = \frac{1}{4}(5y + 3)$$

then interchanging the variables x and y gives

$$y = f^{-1}(x) = \frac{1}{4}(5x + 3)$$

In turns out that for all bijective functions, ie functions that have the property that they are injective and surjective, have an inverse. The functions we considered above have this property. Sometimes the domain and codomain of the functions are such that the function is not injective and surjective and it is necessary to redefine the codomain and domain so as to construct a bijective function which has an inverse. We will discuss this further in the context of circular functions.

1.3.2 Composite functions

In many practical applications the mathematical model will involve several different functions. For example, the kinetic energy of a moving particle is a function of velocity, so that $T = F(v)$. Also the velocity is a function of time $v = g(t)$, by eliminating v we have

$$T = f(v) = f(g(t)) \tag{1.8}$$

A function of the form $y = f(g(x))$ is called a **function of a function** or a **composite** of the function $f(x)$ and $g(x)$. In modern mathematics text books it is is common to denote this as $f \circ g$.

Note that in general the composition of functions is in general not commutative

$$f(g(x)) \neq g(f(x)) \tag{1.9}$$

Example
If $f(x) = x^2 + 2x$ and $y = g(x) = x + 1$ obtain the composite functions $f(g(x))$ and $g(f(x))$
Solution
To obtain the composite function $f(g(x))$ replace x in the expression for $f(x)$ by $g(x)$ giving

$$\begin{aligned} f(g(x)) &= (g(x))^2 + 2g(x) \\ &= (x+1)^2 + 2(x+1) = x^2 + 2x + 1 + 2x + 2 \\ &= x^2 + 4x + 3 \end{aligned}$$

To obtain the composite function $g(f(x))$ replace x in the expression for $g(x)$ by $f(x)$ giving

$$\begin{aligned} g(f(x)) &= f(x) + 1 \\ &= x^2 + 2x + 1 \end{aligned}$$

clearly $f(g(x)) \neq g(f(x))$.

1.4 Summary and Further Reading

The key topics from this chapter and references to further reading in James (customised edition), James (fourth edition) and Croft and Davison (third edition) are

- Real and integer numbers and the number line. Croft and Davison [pg3, 54-57]. James (customised and fourth edition) [pg 2]
- Revision of fractions. Croft and Davison [pg 17-31].
- Fundamentals of arithmetic, algebra. Croft and Davison [pg 3-14, 55-129, 200-228]. James (customised and fourth edition) [pg3-6, 12-37].
- Open and closed intervals, Modulus. Croft and Davison [pg 55]. James (customised and fourth edition) [pg 8].
- Inequalities. Croft and Davison [pg 238-247]. James (customised and fourth edition) [pg 7-11].
- Functions (Introduction to). Croft and Davison [pg 132-157]. James (customised and fourth edition) [pg 64-82].

Chapter 2

Elementary Functions

In this chapter we will learn about a sequence of elementary functions including

- Polynomial functions
- Rational functions
- Circular functions
- Exponential, logarithmic and hyperbolic functions

These functions often arise in all areas of engineering and it is important to have a good understanding of how they operate.

2.1 Polynomial Functions

A **polynomial function** has the general form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{2.1}$$

where n is the a positive integer, a_r is a real number called the **coefficient** of x_r, $r = 0, 1, \cdots, n$. The index n of the highest power of x occurring in $f(x)$ is called the **degree**. For $n = 1$ we obtain the linear function

$$f(x) = a_1 x + a_0 \tag{2.2}$$

for $n = 2$ we get the quadratic function

$$f(x) = a_2 x^2 + a_1 x + a_0 \tag{2.3}$$

and so on. Polynomials have two important properties

1. If two polynomials are equal for all values of the independent variable, then coefficients of the powers of the variable are equal. Thus if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{2.4}$$
$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0 \tag{2.5}$$

   and $f(x) = g(x)$ for all x then $a_i = b_i$ for $i = 0, 1, \cdots, n$.

2. Any polynomial with real coefficients can be expressed as a product of **linear and irreducible quadratic factors** (an irreducible quadratic factor is one that cannot be factored into the product of two linear terms with real coefficients).

Example
Find values of A, B and C to ensure that $x^2 + 1 = A(x-1) + B(x+2) + C(x^2+2)$ for all values of x.
Solution
We first gather the appropriate terms together on the right hand side

$$x^2 + 1 = Cx^2 + (A+B)x + (-A + 2B + 2C)$$

By comparing and equating coefficients of x^2, x^1 and x^0 we find

$$C = 1 \qquad A + B = 0 \qquad -A + 2B + 2C = 1$$

which lead to the result

$$A = \frac{1}{3} \qquad B = -\frac{1}{3} \qquad C = 1$$

Example
Factorise the polynomial $x^3 - 3x^2 + 6x - 4 = 0$
Solution
Clearly the function $f(x) = x^3 - 3x^2 + 6x - 4$ is equal to zero when $x = 1$. We say that $x = 1$ is a **zero** of the function. Thus $x - 1$ must be a factor of $f(x)$. We factor out $x - 1$ as follows

$$x^3 - 3x^2 + 6x - 4 = (x-1)(x^2 - 2x + 4)$$

We now have $(x-1)$ multiplied by $g(x) = (x^2 - 2x + 4)$. To see if we can simplify this further, we need to find the zeros of $g(x)$. The zeros of $g(x)$ are the roots of the quadratic equation $(x^2 - 2x + 4) = 0$ which we can find using equation (1.5)

$$x = \frac{2 \pm \sqrt{4 - 4(4)}}{2}$$

In this case we see that are no real roots as $\sqrt{4-16} = \sqrt{-12}$ is not defined. Therefore $(x^2 - 2x + 4)$ is an irreducible quadratic term and our factorised solution is

$$x^3 - 3x^2 + 6x - 4 = (x-1)(x^2 - 2x + 4)$$

2.2 Rational Functions

Rational functions have the general form

$$f(x) = \frac{p(x)}{q(x)} \tag{2.6}$$

where $p(x)$ and $q(x)$ are polynomials. If the degree of p is less than the degree of q it is known as **proper** otherwise it is known as **improper**. An improper rational function can always be expressed as a polynomial plus a rational function, e.g.

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1} \tag{2.7}$$

A proper rational function can always be expressed as a sum of simpler functions whose denominator are linear or quadratic irreducible factors, e.g.

$$\frac{x^2+1}{(1+x)(1-x)(2+2x+x^2)} = \frac{1}{1+x} + \frac{1}{5(1-x)} + \frac{4x+7}{5(2+2x+x^2)} \tag{2.8}$$

these functions are called **partial functions** of the rational function and are often useful in the mathematical analysis of engineering systems.

2.2.1 Summary of the Method

In general the method for finding partial fractions of a given function $f(x) = \frac{p(x)}{q(x)}$ consists of the following steps

1. Factorise $q(x)$ fully into linear and irreducible quadratic factors, collecting together all like factors.

2. Each *linear* factor $ax+b$ in $q(x)$ will give rise to a fraction of the type

$$\frac{A}{ax+b} \tag{2.9}$$

   where A remains to be found. If there are repeated linear factor $(ax+b)^n$, they will give rise to n fractions of the type

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n} \tag{2.10}$$

   where $A_1, A_2, \cdots, A_n$ remain to be found. Each irreducible quadratic factor ax^2+bx+c in $q(x)$ will give rise to a fraction of the type

$$\frac{Cx+D}{ax^2+bx+c} \tag{2.11}$$

   where C and D remain to be found. Sum all fractions together.

3. If the degree of $p(x)$ is n and the degree of $q(x)$ is m and $n \geq m$ then the function is improper and an additional polynomial of the form $B_1 + B_2x + \cdots$ of degree $n-m$ must be added to the sum of factors.

4. Put $\frac{p(x)}{q(x)}$ equal to the sum of all the factors involved.

5. Multiply both sides of the equation by $q(x)$ to obtain an identity involving a polynomial on both the left and right hand side of the equals sign. The multiplying constants may be found from this identity.

6. To find the coefficients the following technique may be used: Compare the coefficients of like powers of x on both sides of the identity. Starting with highest and working towards the lowest power usually makes it easier.

7. Check the result by using a test value for x.

Example
Express $\frac{3x}{(x-1)(x+2)}$ as a partial fraction.
Solution
We first write the rational fraction as a sum of simple fractions

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

We then multiply both the top and bottom by $(x-1)(x+2)$ and gather terms in x^1 and x^0.

$$3x = A(x+2) + B(x-1) = (A+B)x + (2A - B)$$

comparing coefficients of powers of x gives

$$3 = A + B \qquad 0 = 2A - B$$

and gives $A = 1$ and $B = 2$ so that

$$\frac{3x}{(x-1)(x+2)} = \frac{1}{(x-1)} + \frac{2}{(x+2)}$$

Example
Express $\frac{3x^2}{(x-1)(x+2)}$ as a partial fraction.
Solution
In this case the numerator is the same degree as the denominator so we write

$$\frac{3x^2}{(x-1)(x+2)} = A + \frac{B}{(x-1)} + \frac{C}{(x+2)}$$

Multiplying both sides by $(x-1)(x+2)$ and gathering similar terms yields

$$3x^2 = A(x-1)(x+2) + B(x+2) + C(x-1) = Ax^2 + (B+C+A)x + (-2A+2B-C)$$

comparing coefficients of powers of x gives

$$3 = A \qquad 0 = B + C + A \qquad 0 = -2A + 2B - C$$

which results in $A = 3$, $B = 1$ and $C = -4$ and

$$\frac{3x^2}{(x-1)(x+2)} = 3 + \frac{1}{(x-1)} - \frac{4}{(x+2)}$$

2.3 Circular Functions

There are two approaches to the definition of **circular** or **trigonometric functions**, one approach is static the other is dynamic. The **static** approach began with practical problems of surveying and gave rise to triangles and measurement we call trigonometry. We consider a right angle triangle ABC, where $\angle CAB$ is the right–angle. Thus using Figure 2.1 we have

Figure 2.1: Right angled triangle

Figure 2.2: General triangle

$$\sin\theta = \frac{c}{a} = \frac{\text{opposite}}{\text{hypotenuse}} \tag{2.12}$$

$$\cos\theta = \frac{b}{a} = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{2.13}$$

$$\tan\theta = \frac{c}{b} = \frac{\text{opposite}}{\text{adjacent}} \tag{2.14}$$

This is extended to non –right angled triangles by using the sine and cosine rules

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{Sine Rule} \tag{2.15}$$

$$\left.\begin{array}{l} a^2 = b^2 + c^2 - 2bc\cos A \\ b^2 = a^2 + c^2 - 2ac\cos B \\ c^2 = a^2 + b^2 - 2ab\cos C \end{array}\right\} \qquad \text{Cosine Rule} \tag{2.16}$$

where the side lengths and angles are as displayed in Figure 2.2. The **dynamic** definition of the functions arises from considering the motion of a point around a circle.

It is common in many aspects of engineering to measure angles in radians rather than degrees. As standard we write

$$180^o = \pi\text{radians} \tag{2.17}$$

which we use to convert degrees to radians and vice versa. For angles measured in degrees it is possible to compute the arc and segment of a circle are shown in Figure 2.3. These may be computed as

$$\text{Arc of circle} = r\theta \tag{2.18}$$

$$\text{Area of segment} = \frac{1}{2}r^2\theta \tag{2.19}$$

but only when θ is the angle in radians.

(a) (b)

Figure 2.3: Illustration of (a) an arc and (b) the segment of a circle

(a) (b)

Figure 2.4: Graphs of $\sin\theta$ for (a) $-2\pi \leq \theta \leq 0$ and (b) $0 \leq \theta \leq 2\pi$

The graph of $\sin\theta$ shown in Figure 2.4. replicates itself at intervals of 2π radians (or 360 degrees) we can write

$$\sin(\theta + 2\pi k) = \sin\theta \qquad k = 0, \pm 1, \pm 2 \tag{2.20}$$

$\sin\theta$ is said to be **periodic** with **period** 2π. The function $f(x) = \sin x$ is an example of an **odd function**, which has the property $f(x) = -f(-x)$ for all x, ie $\sin x = -\sin(-x)$.

The graph of $\cos\theta$ is shown in Figure 2.5. Again the function is periodic with period 2π

$$\cos(\theta + 2\pi k) = \cos\theta \qquad k = 0, \pm 1, \pm 2 \tag{2.21}$$

The function $f(x) = \cos x$ is an example of an **even function**, which has the property $f(x) = f(-x)$ for all x, ie $\cos x = \cos(-x)$.

The graph of $\tan\theta$ is shown in Figure 2.6.

Again this function is periodic, but now with period π

$$\tan(\theta + \pi k) = \tan\theta \qquad k = 0, \pm 1, \pm 2, \cdots \tag{2.22}$$

The function $f(x) = \tan x$ is an odd function.

It can be shown that the definitions of sine, cosine and tangent are associated with the properties of a circle and so are called **circular functions**. We summarise the functions that are positive in respective quadrants of the circle by using the diagram shown in Figure 2.7.

Figure 2.5: Graphs of $\cos\theta$ for (a) $-2\pi \le \theta \le 0$ and (b) $0 \le \theta \le 2\pi$

Figure 2.6: A graph of $\tan\theta$ for $0 \le \theta \le 2\pi$

2.3.1 Inverse circular functions

Inverse sine function

The domain and the codomain of the sine function $y = f(x) = \sin x$ are $\mathbb{R}$. The function is neither injective nor surjective. Its range is the set

$$V = \{y \in \mathbb{R} : -1 \le y \le 1\} \tag{2.23}$$

In order to set about defining a function which is bijective we first cut the codomain from $\mathbb{R}$ to V. Having done this, there remains an infinite number of ways in which the domain of f can be cut down to make f a bijective function with codoamin V. It is customary to cut down the size

Figure 2.7: Diagram showing in which quadrant the circular functions are positive

of the domain of f from $\mathbb{R}$ to

$$U = \left\{x \in \mathbb{R} : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\} \tag{2.24}$$

The graph of $y = f(x) = \sin x$ for this bijective function is shown in Figure 2.8 (left).

For the above bijective function there exists an inverse function. This inverse function is known as the **inverse sine function** and is denoted by $\sin^{-1}$ or arcsin. The graph of the inverse sine function is shown in Figure 2.8 (right). Note that $\sin^{-1}$ is defined only for real numbers between -1 and 1 inclusive. It takes all values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ exactly once but does not take any values outside that closed interval.

$\sin x$ $\qquad\qquad$ $\sin^{-1} x$

Figure 2.8: Diagram showing the $y = \sin x$ with domain $x \in U$ and codomain $y \in V$ and its corresponding inverse function $y = \sin^{-1} x$

Inverse cosine function

The domain and the codomain of the cosine function $y = f(x) = \cos x$ are $\mathbb{R}$. Just as with the sine function, this function is neither injective or surjective. Its range is the set

$$V = \{y \in \mathbb{R} : -1 \leq y \leq 1\} \tag{2.25}$$

We first cut down the codomain from $\mathbb{R}$ to V, In the case of the cosine function, it is customary to cut down the size of the domain of f from $\mathbb{R}$ to

$$W = \{x \in \mathbb{R} : 0 \leq x \leq \pi\} \tag{2.26}$$

The graph of $y = f(x) = \cos x$ for this bijective function is shown in Figure 2.9 (left).

For the above bijective function there exists an inverse function. This inverse function is known as the **inverse cosine function** and is denoted by $\cos^{-1}$ or arccos. The graph of the inverse cosine function is shown in Figure 2.9 (right). Note that $\cos^{-1}$ is defined only for real numbers between -1 and 1 inclusive. It takes all values between 0 and π exactly once but does not take any values outside that closed interval.

Inverse tangent function

The codomain of the tangent function $y = f(x) = \cos x$ is $\mathbb{R}$. However its domain is

$$X = \left\{x \in \mathbb{R} : x \neq (2k+1)\frac{\pi}{2} \text{ where } k \in \mathbb{Z}\right\} \tag{2.27}$$

$\cos x$ $\cos^{-1} x$

Figure 2.9: Diagram showing the $y = \cos x$ with domain $x \in W$ and codomain $y \in V$ and its corresponding inverse function $y = \cos^{-1} x$

$\tan x$ $\tan^{-1} x$

Figure 2.10: Diagram showing the $y = \tan x$ with domain $x \in S$ and codomain $y \in \mathbb{R}$ and its corresponding inverse function $y = \tan^{-1} x$

This function is surjective but not injective. As the range is the same as the codomain of the function and we only need to cut down the size of the domain from X to

$$S = \left\{ x \in \mathbb{R} : -\frac{\pi}{2} < x < \frac{\pi}{2} \right\} \tag{2.28}$$

The graph of $y = f(x) = \tan x$ for this bijective function is shown in Figure 2.10 (left).

For the above bijective function there exists an inverse function. This inverse function is known as the **inverse tangent function** and is denoted by $\tan^{-1}$ or arctan. The graph of the inverse tangent function is shown in Figure 2.10 (right). Note that $\tan^{-1}$ is defined for all real numbers. It takes all values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ but not $-\frac{\pi}{2}$ or $\frac{\pi}{2}$ themselves and also does not take values outside this open interval.

2.3.2 Identities for circular functions

Other circular functions are defined in terms of these basic functions sine, cosine and tangent. In particular we have the following three functions

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \tag{2.29}$$

Note that these functions are quite different from the inverse functions and in particular $\sec\theta \neq \cos^{-1}\theta$, cosec$\theta \neq \sin^{-1}\theta$ and $\cot\theta \neq \tan^{-1}\theta$. We state also the triangle identities

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1 && (2.30)\\ 1 + \tan^2 x &= \sec^2 x && (2.31)\\ 1 + \cot^2 x &= \text{cosec}^2\, x && (2.32) \end{aligned}$$

and the compound angle identities

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \cos x \sin y && (2.33)\\ \sin(x-y) &= \sin x\cos y - \cos x \sin y && (2.34)\\ \cos(x+y) &= \cos x\cos y - \sin x \sin y && (2.35)\\ \cos(x-y) &= \cos x\cos y + \sin x \sin y && (2.36)\\ \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} && (2.37)\\ \tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x \tan y} && (2.38)\\ & && (2.39) \end{aligned}$$

In addition the sum and product identities

$$\begin{aligned} \sin x + \sin y &= 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) && (2.40)\\ \sin x - \sin y &= 2\sin\frac{1}{2}(x-y)\cos\frac{1}{2}(x+y) && (2.41)\\ \cos x + \cos y &= 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) && (2.42)\\ \cos x - \cos y &= -2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y) && (2.43) \end{aligned}$$

are sometimes also useful.

Example
Show that $\sin 2\theta = 2\sin\theta\cos\theta$
Solution
By using $\sin(x+y) = \sin x\cos y + \cos x\sin y$ with $x = y = \theta$ we have

$$\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$$

Example
Solve $2\sin^2 x - 3\sin x + 1 = 0$ for $0 \leq x \leq 2\pi$
Solution
We first recognise that this is quadratic equation in $\sin x$. Writing $\lambda = \sin x$ we have

$$2\lambda^2 - 3\lambda + 1 = 0$$

The roots of this quadratic equation are given by $\lambda = \frac{3\pm\sqrt{3^2-4(2)}}{4} = 1,\ \frac{1}{2}$
i) When $\lambda = 1$ then $\sin x = 1$ and we have the solution $x = \frac{\pi}{2}$.
ii) When $\lambda = \frac{1}{2}$ then $\sin x = \frac{1}{2}$. Remembering that $\sin x$ is positive in the first and second quadrants we have the solutions $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

Figure 2.11: Graphs of a selection of exponential functions

Example
Express $y = 4\sin 3t - 3\cos 3t$ in the form $y = A\sin(3t + \alpha)$
Solution
We use the identity $\sin(x + y) = \sin x \cos y + \sin y \cos x$ and write

$$A\sin(3t + \alpha) = A\sin 3t \cos\alpha + A\sin\alpha\cos 3t$$

We also know that

$$y = 4\sin 3t - 3\cos 3t = A\sin 3t\cos\alpha + A\sin\alpha\cos 3t$$

leading to the two conditions $A\cos\alpha = 4$ and $A\sin\alpha = -3$. Squaring and adding these equations gives

$$A^2\cos^2\alpha + A^2\sin^2\alpha = A^2 = 16 + 9 = 25$$

Thus $A = \pm 5$. Choosing $A = 5$ and dividing the the two previously found conditions gives

$$\frac{A\sin\alpha}{A\cos\alpha} = \tan\alpha = -\frac{3}{4}$$

Now as we have chosen A to be positive, $\cos\alpha$ must be positive and $\sin\alpha$ must be negative and hence α lies in the fourth quadrant. Specifically it is given by $\alpha = -0.64rad$ and thus

$$y = 5\sin(3t - 0.64)$$

2.4 Exponential, Logarithmic and Hyperbolic Functions

Functions of the type $f(x) = a^x$ where a is a constant (and x is the independent variable) are called **exponential functions**. Exponential functions occur widely in engineering, applications range from heat transfer problems to the design of nuclear reactors. The graphs of exponential functions are similar, as can be seen from Figure 2.11. The reason for this is that we can write $y = 4^x = 2^{2x}$ etc. It follows that all exponential functions can be written in terms of a single exponential function. The standard exponential function that is used is $y = e^x$ where e is a special number approximately equal to

$$e = 2.71828128845 \tag{2.44}$$

This number is chosen since the graph $y = e^x$ has the property that the slope of the tangent at any point of the curve is equal to the value of the function at the point. The domain and codomain of this function is $\mathbb{R}$ and its range is $\mathbb{R}^+$ (the set of real positive numbers). This exponential function has the following properties

$$e^{x_1}e^{x_2} = e^{x_1+x_2} \tag{2.45}$$
$$e^{x+c} = e^x e^c = Ae^x \qquad \text{where } A = e^c \tag{2.46}$$
$$\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2} \tag{2.47}$$
$$e^{kx} = (e^k)^x = a^x \qquad \text{where } a = e^k \tag{2.48}$$

The exponential function is clearly injective, however it is not surjective. In order to define its inverse, we first need to make it an injective function, to do this we cut down the codomain and make it equal to the range of the function $\mathbb{R}^+$. This redefinition of the codomain makes the function bijective so that the inverse function is defined. Its inverse is known as the **natural logarithm**

$$y = \ln x \tag{2.49}$$

Note that some advanced mathematics text books may refer to this as $\log x$. If $y = e^x$ then $x = \ln y$ which implies that $\ln e^x = x$ and $e^{\ln y} = y$.

In the same way that there are many exponential functions ($2^,3^x, 4^x$ etc) there are many logarithmic functions. In general

$$y = a^x \qquad \text{gives } x = \log_a y \tag{2.50}$$

where $a > 1$ or $0 < a < 1$ is the base of the logarithm. Note that $\log_{10}$ is frequently referred to as simply $\log x$ (except in advanced modern mathematics textbooks). It follows that

$$\log_a(x_1x_2) = \log_a x_1 + \log_a x_2 \tag{2.51}$$
$$\log_a\left(\frac{x_1}{x_2}\right) = \log_a x_1 - \log_a x_2 \tag{2.52}$$
$$\log_a x^n = n\log_a x \tag{2.53}$$
$$x = a^{\log_a x} \tag{2.54}$$
$$y^x = a^{x\log_a y} \tag{2.55}$$
$$\log_a x = \frac{\log_b x}{\log_b a} \tag{2.56}$$
$$\tag{2.57}$$

The natural logarithm is the case where $a = e$ so that $\log_e x = \ln x$, and thus the above rules also apply to natural logarithms.

Associated with the exponential functions is a family of functions called the **hyperbolic functions**. They are defined as follows

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} \tag{2.58}$$

the reason for these names is geometric. They bear the same relationship to a hyperbola as the circular functions do to the circle. The graphs of the hyperbolic functions are shown in Figure 2.12.

From these graphs we can observe that the $y = \cosh x$ function has domain and codomain $\mathbb{R}$ and range $T = \{y \in \mathbb{R} : y \geq 1\}$. It can be made surjective by restricting the codomain to the range and injective by choosing $x \geq 0$. The $y = \sinh x$ function has domain, codomain and range

Figure 2.12: Graphs of $\sinh x$, $\cosh x$ and $\tanh x$

$\mathbb{R}$. Hence it is injective and surjective. The $y = \tanh x$ function has domain and codomain $\mathbb{R}$ but its range is $-1 < y < 1$. The function is injective but not surjective. It can be made surjective by choosing the codomain to be the same as the range.

The inverse hyperbolic functions can then be defined in a natural way

$$x = \cosh^{-1} y \qquad y \geq 1, x \geq 0 \tag{2.59}$$
$$x = \sinh^{-1} y \qquad y \text{ in } \mathbb{R} \tag{2.60}$$
$$x = \tanh^{-1} y \qquad -1 < y < 1 \tag{2.61}$$

where the restrictions on x and y are required in order to make the function bijective (injective and surjective).

Other hyperbolic functions are defined as

$$\text{sech}x = \frac{1}{\cosh x} \tag{2.62}$$
$$\text{cosech}x = \frac{1}{\sinh x} \qquad (x \neq 0) \tag{2.63}$$
$$\text{coth}x = \frac{1}{\tanh x} \qquad (x \neq 0) \tag{2.64}$$

The hyperbolic functions satisfy a series of relationships which include

$$\cosh x + \sinh x = e^x \tag{2.65}$$
$$\cosh x - \sinh x = e^{-x} \tag{2.66}$$
$$(\cosh x + \sinh x)(\cosh x - \sinh x) = e^x e^{-x} \tag{2.67}$$
$$\cosh^2 x - \sinh^2 x = 1 \tag{2.68}$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{2.69}$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \tag{2.70}$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \tag{2.71}$$

Example
Solve the equation $5\cosh x + 3\sinh x = 4$
Solution
We first express the hyperbolic functions in terms of exponential functions and simplify

$$\begin{aligned}\frac{5}{2}(e^x + e^{-x}) + \frac{3}{2}(e^x - e^{-x}) &= 4 \\ 4e^x - 4 + e^{-x} &= 0 \\ 4(e^x)^2 - 4e^x + 1 &= 0\end{aligned}$$

where the last equation resulted by multiplying both sides of the equation by e^x. This is quadratic equation in e^x, which we can solve using(1.5) giving

$$e^x = \frac{4 \pm \sqrt{16 - 4(4)}}{8} = \frac{1}{2}$$

Thus $e^x = \frac{1}{2}$ and $x = \ln\frac{1}{2}$

2.5 Continuous and Discontinuous Functions

Let us consider the function $\cos x$ shown in Figure 2.13. Clearly the function can be traced from left to right without removing pen from paper. Such a function is called **continuous function**. Next let us consider the two functions shown in Figure 2.13, these functions cannot be drawn without lifting the pen from paper and are called **discontinuous functions**. The jumps in these functions are called **discontinuities**.

Figure 2.13: A graph of $\cos x$ which is an example of a continuous function

The first of these two functions is called the **signum function** and is defined as

$$\text{sgn}\, x = \begin{cases} +1 & (x > 0) \\ -1 & (x < 0) \\ 0 & (x = 0) \end{cases} \tag{2.72}$$

and the second of these two functions is called the **Heaviside unit step function**, it is defined by

$$H(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x \geq 0) \end{cases} \tag{2.73}$$

$\operatorname{sgn} x$ $\qquad$ $H(x)$

Figure 2.14: Graphs of $\operatorname{sgn} x$ and $H(x)$ which are examples of discontinuous functions

To describe continuous and discontinuous functions mathematically requires an understanding of the **limit** of a function. This is best achieved through use of an example. Let us consider the graph of the linear function $y = f(x) = 6x - 5$ as shown in Figure 2.15. From this figure we

Figure 2.15: A graph of $y = f(x) = 6x - 5$

observe that we x approaches 3, y approaches 13. We write this concisely as

$$y \to 13 \qquad \text{as } x \to 3 \tag{2.74}$$

or more commonly as

$$\lim_{x \to 3} y = 13 \tag{2.75}$$

Similarly we see that as x approaches -2, y approaches -17 which we write as

$$\lim_{x \to -2} y = -17 \tag{2.76}$$

For the function $f(x) = 6x - 5$ it doesn't matter whether our chosen values of x are approached from the left or the right. But this is not true for all functions.

Let us consider the graph of the function

$$y = f(x) = \begin{cases} 2x + 1 & x < 3 \\ 5 & x = 3 \\ 6 & x > 3 \end{cases} \tag{2.77}$$

which we show in Figure 2.16. If x approaches 3 from the left and right we obtain different

Figure 2.16: A graph of the discontinuous function $f(x)$ illustrating the effect of left and right hand limits

answers. From the left we write

$$\lim_{x \to 3^-} y = 7 \tag{2.78}$$

and say that the **left-hand limit** of y is 7. When x approaches 3 from the right we write

$$\lim_{x \to 3^+} y = 6 \tag{2.79}$$

and we say that the **right-hand limit** of y is 6.

Example
For the function

$$f(x) = \begin{cases} x+1 & x \neq 2 \\ 5 & x = 2 \end{cases}$$

compute $a)$ $\lim_{x\to 2^-} f(x)$, $b)$ $\lim_{x\to 2^+} f(x)$, $c)$ $\lim_{x\to 4-} f(x)$ and $d)$ $\lim_{x\to 4^+} f(x)$.
Solution
$a)$ If we examine the function we have $\lim_{x\to 2^-} f(x) = 3$
$b)$ Examining the function we have $\lim_{x\to 2^+} f(x) = 3$. Now as $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 3$ then $\lim_{x\to 2} f(x) = 3$. Note that $\lim_{x\to 2} f(x) \neq f(2)$.
$c)$ If we examine the function we have $\lim_{x\to 4^-} f(x) = 5$
$d)$ Examining the function we have $\lim_{x\to 4^+} f(x) = 5$. Now again as $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = 5$ then $\lim_{x\to 4} f(x) = 5$ and in this case $\lim_{x\to 4} f(x) = f(4) = 5$

2.6 Summary and Further Reading

The key topics from this chapter and references to further reading in James (customised edition), James (fourth edition) and Croft and Davison (third edition) are

- Polynomial functions. Croft and Davison [pg 171-175, 220-228]. James (customised and fourth edition) [pg 87-114]
- Rational functions. Croft and Davison [pg. 184-188, 248-258]. James (customised and fourth edition) [pg 114-121]
- Circular functions. Croft and Davison [pg 313-420]. James (customised and fourth edition) [pg 128-151].

- Exponential and logarithmic functions. Croft and Davison [pg. 267-310]. James (customised and fourth edition) [152-164].

- Continuous and discontinuous functions. Croft and Davison [pg 161-163, 189-192]. James (customised and fourth edition) [pg. 170-173]

Chapter 3

Introduction to Complex Numbers

The numbers we have encountered so far within these notes have been real numbers. In order to solve certain types of mathematical problems it is necessary to introduce further numbers. These numbers are called complex numbers. An important application of complex numbers is in the analysis of alternating current circuits. In this chapter we shall introduce some of the properties of complex numbers, we shall revisit them in more detail in EG190.

3.1 The Number j

We know that when we square a positive or negative number the result is always positive, for example $3^2 = 9$ and $(-3)^2 = 9$. Let us now suppose that we would like to determine $\sqrt{-9}$, unfortunately the mathematics we have learnt up until now will not allow us to perform this operation, as the square root only makes sense for positive real numbers. For certain applications it is useful to overcome this limitation. To do this we introduce a new number, to which we will give the symbol j, which has the property that

$$j^2 = -1 \qquad \text{so that } j = \sqrt{-1} \tag{3.1}$$

As no real number when squared equals -1, the number j cannot be real. Instead we call it an **imaginary** number. Although the concept of an imaginary number may seem strange at first, in turns out to be very useful in engineering applications. Mathematicians and physicists often prefer the symbol i instead of the symbol j for $\sqrt{-1}$.

Using this notation we now in a position to write down the square root of any negative number, for example $\sqrt{-9} = j3$, the result is not a real number, but instead an imaginary number.

3.2 The Complex Number $a + jb$

In the first chapter we derived the formula for obtaining general roots to the quadratic equation $ax^2 + bx + c = 0$, namely

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.2}$$

As soon as $4ac > b^2$, the term $b^2 - 4ac$ will become negative and thus the above expression will involve the square root of a negative number. In this case we can use j to help us evaluate the square root. For example the quadratic equation $x^2 - 6x + 10 = 0$ has the roots $x = 3 \pm j$. Thus we can write down the solutions of the equation as $3 + j$ and $3 - j$, these two numbers are called **complex numbers**. Each number consists of two parts: a **real part** and an **imaginary part**. The set of all complex numbers is given the symbol $\mathbb{C}$.

(a) (b)

Figure 3.1: Argand diagrams of complex numbers showing, (a) a selection of complex numbers with the point A representing $3 + j3$ and B representing $-1 + j$, and (b) the polar form of the complex number $z = x + jy$

In general we give complex numbers the symbol z and write them as

$$z = x + jy \tag{3.3}$$

where $x = \text{Re}(z)$ is the real part of the complex number and $y = \text{Im}(z)$ is its imaginary part. When the complex number is written in this way it is called its **Cartesian form**.

Example

Determine the roots of the quadratic equation $x^2 - 3x + 4 = 0$

Solution

Applying equation (1.5) we have

$$x = \frac{3 \pm \sqrt{9 - 16}}{2} = \frac{3 \pm \sqrt{-7}}{2} = \frac{3 \pm j\sqrt{7}}{2} \tag{3.4}$$

here we used $j = \sqrt{-1}$ to rewrite $\sqrt{-7}$ as $j\sqrt{7}$.

3.3 Graphical Representation

Complex numbers can be represented as points on a plane in a similar way to which real numbers are represented by points on a curve. The number $z = x + jy$ is represented by the point P with coordinates (x, y). Figure 3.1 (a) shows a sequence of complex numbers and their graphical representations. Such a diagram is called an **Argand diagram** after one of its inventors. The x axis is called the **real axis** and the y axis is called the **imaginary axis**.

Following the introduction of the Argand diagram we now have another method of specifying a complex number. As indicated in Figure 3.1 (b), the point P is uniquely determined if we know the length of the line OP and the angle it makes with the x axis. The length OP is a measure of the size of z, and is called the **modulus** of z, which is usually written as denoted by mod$z$ or $|z|$. The angle between the real axis and OP is called the **argument** of z, and is denoted by $\arg z$. Note that the polar coordinates (r, θ) and $(r, \theta + 2\pi)$ represent the same point, however, a convention is adopted to determine the argument of z uniquely: We restrict its range so that $-\pi \leq \theta \leq \pi$. The argument of the complex number $0 + j0$ is not defined.

Thus from Figure 3.1 (b) $|z|$ and $\arg x$ are given by

$$|z| = r = \sqrt{x^2 + y^2} \tag{3.5}$$

$$\arg z = \theta \qquad \text{where } \tan\theta = \frac{y}{x} \tag{3.6}$$

Care must be taken to ensure that $\arg z$ is computed for the correct quadrant. By plotting the complex number in the Argand diagram one can ensure that the result makes sense.

3.4 Polar Form of a Complex Number

From Figure 3.1 (b) we easily obtain the relationships between (x, y) and (r, θ)

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \tag{3.7}$$

It therefore follows that the complex number $z = x + jy$ can be expressed in the form

$$z = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta) \tag{3.8}$$

This is called the **polar form** of the complex number and is frequently written as $r\angle\theta$

$$z = r\angle\theta = r(\cos\theta + j\sin\theta) \tag{3.9}$$

Example
Express $-4 - j$ in polar form.
Solution
First, we sketch the Argand diagram which shows the location of $z = -4 - j$

$$|z| = \sqrt{(-4)^2 + (-1)^2} = \sqrt{16 + 1} = \sqrt{17} \tag{3.10}$$

$$\arg z = -\pi + \tan^{-1}\frac{1}{4} = -2.89 \text{ (2dp) Radians} \tag{3.11}$$

Thus the polar form of the number is $z = \sqrt{17}(\cos -2.89 + j\sin -2.89) = \sqrt{17}(\cos 2.89 - j\sin 2.89)$

3.5 Summary and Further Reading

The key topics from this chapter and references to further reading in James (fourth edition), James (customised edition) and Croft and Davison (second edition) are

- The complex number $a + jb$ Croft and Davison [pg 424-429]. James (customised and fourth edition) [pg 185-186]

- Argand diagram. Croft and Davison [pg 435-443]. James (customised and fourth edition) [pg 191-196].
- Polar from. Croft and Davison [pg. 443-449]. James (customised and fourth edition) [pg. 196-200].

Chapter 4

Differentiation

Many of the practical situations that engineers have to analyse involve quantities that are varying. Whether it is the temperature of a coolant, the voltage of a transmission line or the torque on a turbine blade, the mathematical tools for performing such analyses are the same. One of the most used tools is **calculus** which involves two main operations integration and differentiation. Historically integration was discovered first in relation to trying to find the area of a region bounded on one side by a curve. Differentiation was discovered during the 17th century in relation to the problem of determining the tangent at any arbitrary point on a curve.

The connection between the two processes of determining the area under a curve and obtaining a tangent at some point on a curve was first realised in 1663 by Barrow, who was Newton's professor at Cambridge. However it was Newton and Leibnitz, working independently who fully realised the implications of this relationship. They developed calculus to be a way of dealing with change and motion and applied it to many practical problems. Calculus remains today one of the most powerful mathematical tools used by engineers. In this chapter we shall look in more detail at differentiation and the in the next chapter we will consider integration.

4.1 Basic Ideas and Definitions

To help us to understand the concept of differentiation let us first begin with a basic physics example. We consider an object moving along a line with constant velocity u (in ms^{-1}). The distance s (in m) travelled by the object in time t (in s) is given by $s = ut$. The distance time graph is shown in Figure 4.1.

We see that the velocity u is just the gradient (slope) of the distance–time graph. This is of course a special case where the velocity is just a constant function. Let us now generalise and consider the case when the velocity varies with time. In this case the velocity is still given by the gradient of this graph although now it obviously varies along the curve.

Figure 4.1: The distance time graph for an object moving with constant velocity in a straight line

(a) (b)

Figure 4.2: The distance time graph for an object moving with variable velocity

Let us consider the distance–time graph shown in Figure 4.2 (a). We can approximate this graph by small straight line segments as shown in Figure 4.2 (b). If we make the time intervals for this approximation very small, the difference between the gradient of the line segments and the gradients of the tangents at each point of the curve become very small. In other words, the velocity at $t = t_1$ for graph (a) is just the gradient of the tangent to the graph at $t = t_1$.

This is one of the many practical problems that involve the process of finding gradients of tangents to graphs in their solution. This process is called **differentiation** and measures the rate of change of the value of the functions with respect to its argument. The gradient of the graph is called the **derivative** of the function. For some functions we can obtain formulae for the derivative while for others we have to remain content just with numerical approximations.

The derivative of a function $f(x)$ at the point x is formally defined as

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \tag{4.1}$$

which uses the $\lim$ operator we saw earlier in Chapter 2. Two kinds of notation are commonly used for representing the derivative. The fist uses a composite symbol

$$\frac{\mathrm{d}f}{\mathrm{d}x} \qquad \text{or} \qquad \mathrm{d}f/\mathrm{d}x \qquad \text{or} \qquad D_f \tag{4.2}$$

The second notation uses a prime

$$f'(x) \tag{4.3}$$

this means that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) = D_f = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{4.4}$$

We illustrate the definition graphically in Figure 4.3, where Δx denotes a small incremental change in the dependent variable x and Δf is the corresponding incremental change in $f(x)$. The slope of the line PQ is

$$\frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{4.5}$$

Now in the limit as $\Delta x \to 0$ the point $P \to Q$ and the segment becomes the tangent to the curve at P, whose slope is given by the derivative

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \tag{4.6}$$

Figure 4.3: Illustration of definition of the derivative

(a) (b)

Figure 4.4: Examples of functions that are not differentiable at certain points

From this interpretation it follows that for a function $f(x)$ to be **differentiable** at the point $x = a$ the graph of $f(x)$ must have a unique, non–vertical well defined tangent at $x = a$. If this is not the case the limit

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \tag{4.7}$$

does not exist. In Figure 4.4 we show examples of functions that are not differentiable at certain points. The function shown in Figure 4.4 (a) is differentiable at all points except $x = x_1$ and $x = x_2$ since a unique tangent cannot be drawn at these points. The function shown in Figure 4.4 (b) is differentiable at all points except $x = 0$. For practical purposes it is sufficient to interpret a differentiable function as one having a smooth continuous graph with no sharp corners. Engineers commonly refer to such functions as being **well–behaved**.

4.2 Elementary Functions

The derivatives of some elementary functions are given in Table 4.1. We have given these functions without proof as they require knowledge of series expansions which will be discussed in the course Engineering Analysis 2. With knowledge of these derivatives we can obtain the derivatives of many other functions, as we will see shortly.

$f(x)$	$f'(x)$
x^n where n is real	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x

Table 4.1: Derivatives of some elementary functions

4.3 Rules of Differentiation

So that we can apply differentiation it is useful to know the following useful rules

- **Rule 1 (scaler multiplication rule)**
  If $y = f(x)$ and k is a constant then

$$\frac{\mathrm{d}}{\mathrm{d}x}(ky) = k\frac{\mathrm{d}y}{\mathrm{d}x} = kf'(x) \tag{4.8}$$

- **Rule 2 (sum rule)**
  If $u = f(x)$ and $v = g(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}x}(u+v) = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}x} = f'(x) + g'(x) \tag{4.9}$$

- **Rule 3 (product rule)**
  If $u = f(x)$ and $v = g(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} = f(x)g'(x) + g(x)f'(x) \tag{4.10}$$

- **Rule 4 (quotient rule)**
  If $u = f(x)$ and $v = g(x)$ then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) - u\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)}{v^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \tag{4.11}$$

- **Rule 5 (composite–function or chain rule)**
  If $z = g(x)$ and $y = f(z)$ then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}x} = f'(z)g'(x) \tag{4.12}$$

- **Rule 6 (inverse–function rule)**
  If $y = f^{-1}(x)$ then $x = f(y)$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y} = \frac{1}{f'(y)} \tag{4.13}$$

By use of the results shown in Table 4.1 and the six rules presented above, one can obtain the derivatives of standard functions.

Example
Find the derivative of $2x^2$
Solution
We use the scalar multiplication rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x^2) = 2\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 4x$$

Example
Find the derivative of $\sin x + x$
Solution
We use the sum rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(x + \sin x) = \frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = 1 + \cos x$$

Example
Find the derivative of $x^3 \sin x$
Solution
We use the product rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^3 \sin x) = \sin x\frac{\mathrm{d}}{\mathrm{d}x}(x^3) + x^3\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = 3x^2 \sin x + x^3 \cos x$$

Example
Find the derivative of $\frac{x^3}{\cos x}$
Solution
We use the quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3}{\cos x}\right) = \frac{\cos x\frac{\mathrm{d}}{\mathrm{d}x}(x^3) - x^3\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)}{\cos^2 x} = \frac{3x^2 \cos x + x^3 \sin x}{\cos^2 x}$$

Example
Find the derivative of $\cos\ \sin x$
Solution
We use the chain rule with $z = \sin x$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\ \sin x\right) = \frac{\mathrm{d}}{\mathrm{d}z}(\cos z)\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = -\sin z \cos x = -\sin\sin x \cos x$$

Example
Find the derivative of $\ln x$.
Solution
If $y = \ln x$ then $x = e^y$ so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = e^y$$

Then, from the inverse–function rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^y} = \frac{1}{x}$$

Many other functions may be found by simple application of elementary results and rules. In Table 4.2 we give the derivative of the most commonly used functions and their derivatives.

$f(x)$		$f'(x)$
x^n	$(n \in \mathbb{R})$	nx^{n-1}
$\sin x$		$\cos x$
$\cos x$		$-\sin x$
$\tan x$		$\sec^2 x$
$\sec x$		$\sec x \tan x$
$\mathrm{cosec}\, x$		$-\mathrm{cosec}\, x \cot x$
$\cot x$		$-\mathrm{cosec}^2\, x$
e^x		e^x
$\ln x$	$(x \in \mathbb{R}^+)$	$\frac{1}{x}$
$\sin^{-1} x$	$(x \in [-1, 1])$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$(x \in [-1, 1])$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$		$\frac{1}{(1+x^2)}$
$\sinh x$		$\cosh x$
$\cosh x$		$\sinh x$
$\tanh x$		$\frac{1}{\cosh^2 x} = 1 - \tanh^2 x$

Table 4.2: Some standard functions and their derivatives

4.4 Parametric and Implicit Differentiation

The composite–function rule is used with the inverse–function rule to evaluate derivatives when a function is specified **parametrically**. In general, if a function is defined by $y = f(x)$, where $x = g(t)$ and $y = h(t)$ and t is parameter then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \tag{4.14}$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} \tag{4.15}$$

The composite function rule may also be used for differentiating functions expressed in an implicit form. For example, let us assume that we have

$$y^3 = x^2 \tag{4.16}$$

and we wish to determine the derivative $\frac{dy}{dx}$. To do this we use a method known as **implicit differentiation**. In this method we treat y as an unknown function of x and differentiate both sides term by term with respect to x. This gives

$$\frac{d}{dx}(y^3) = \frac{d}{dx}(x^2) \tag{4.17}$$

Now y^3 is a composite function of x with y being the intermediate variable, so the composite function rule gives

$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\frac{dy}{dx} = 3y^2\frac{dy}{dx} \tag{4.18}$$

Then substituting back

$$3y^2\frac{dy}{dx} = 2x \qquad \text{or} \qquad \frac{dy}{dx} = \frac{2x}{3y^2} \tag{4.19}$$

4.5 Higher Derivatives

The derivative df/dx of a function $f(x)$ is itself a function and may be differentiable. The derivative of a derivative is called the **second derivative** and is written as

$$\frac{d^2f}{dx^2} \qquad \text{or} \qquad f''(x) \qquad \text{or} \qquad f^{(2)}(x) \qquad \text{or} \qquad D^2f \tag{4.20}$$

To obtain it we simply differentiate df/dx again with respect to x

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \tag{4.21}$$

The second derivative may itself be differentiated, yielding **third derivatives** and so on. In general the n**th derivative** is written as

$$\frac{d^nf}{dx^n} \qquad \text{or} f^{(n)}(x) \qquad \text{or} \qquad D^nf \tag{4.22}$$

Example
Find the second and third derivatives of $y = \sin 2x$
Solution
We first determine the first derivative

$$\frac{dy}{dx} = 2\cos 2x$$

We differentiate this to get the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -4\sin 2x$$

Differentiating again yields the third derivative

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = -8\cos 2x$$

The second derivative d^2f/dx^2 represents the rate of change of df/dx as x increases geometrically, this gives us information on how the slope of the tangent is changing with increasing x.

(a) (b)

Figure 4.5: Implications of $\mathrm{d}^2f/\mathrm{d}x^2$, (a) when $\mathrm{d}^2f/\mathrm{d}x^2 > 0$ and (b) when $\mathrm{d}^2f/\mathrm{d}x^2 < 0$

Figure 4.6: Maximum and minimum values of a function

- If $\mathrm{d}^2f/\mathrm{d}x^2 > 0$ then $\mathrm{d}f/\mathrm{d}x$ is increasing as x increases, and the tangent rotates in an anti-clockwise direction as we move along the horizontal axis, as illustrated in Figure 4.5 (a).
- If $\mathrm{d}^2f/\mathrm{d}x^2 < 0$ then $\mathrm{d}f/\mathrm{d}x$ is decreasing as x increases and the tangent rotates in a clockwise direction as move along the horizontal axis, as illustrated in Figure 4.5 (b).

Note that when $\mathrm{d}^2f/\mathrm{d}x^2 > 0$ the graph of $f(x)$ is always 'concave up' and when $\mathrm{d}^2f/\mathrm{d}x^2 < 0$ the graph of $f(x)$ is always 'concave down'.

4.6 Optimum Values

The basic idea is that the optimum value of a differentiable function $f(x)$ (by which we mean its **maximum** or **minimum value**) generally occurs when its derivative is zero, i.e.

$$f'(x) = 0 \tag{4.23}$$

We can can see this is the case when we observe the graph of a typical function shown in Figure 4.6, since at a maximum or minimum value of the function its graph has a horizontal tangent. What we also observed in Figure 4.6 is that these extremal values are generally only local maximum or minimum values corresponding to turning points on the graph, so some care should be exercised when using the horizontal tangent as a test for an optimal value. In seeking extremal values we need to also check the end points (if any) of the domain of the function.

Another reason to exercise care is at a **point of inflection** as illustrated in Figure 4.7. At this point the graph crosses its own tangent, which as in the illustrated case, may be horizontal.

Figure 4.7: Point of inflection of a function

Figure 4.8: Graph of $f(x) = x^{2/3}$ with minimum at $x = 0$

A third reason for exercising caution when trying to find optimum values is that the function may in fact have an optimal value at a point where the derivative does not exist. A simple example of this is the function $f(x) = x^{2/3}$ illustrated in Figure 4.8.

Having determined the **critical** or **stationary points** where $f'(x) = 0$ we need to determine the character or nature of each of these points. That is we need to determine if these points corresponding to a local maximum, local minimum or point of inflection of the function $f(x)$. One method to do this is the following

- If the value of $f'(x)$ changes from positive to negative as we pass from left to right through a stationary point then the latter corresponds to a **local maximum**.
- If the value of $f'(x)$ changes from negative to positive as we pass from left to right through a stationary point then the latter corresponds to a **local minimum**.
- If $f'(x)$ does not change sign as we pass through a stationary point then the latter corresponds to a point of inflection. In particular we usually call this a **stationary point of inflection** as in this case $f'(x) = 0$ at the point inflection. However points of inflection can occur at other locations too, as we will see shortly.

Another approach to determine the nature of the stationary point is to calculate the value of $f''(x)$ at the point. We recall that $f''(x)$ determines the rate of change of $f'(x)$. Let us suppose that $f(x)$ has a stationary point at $x = a$ so that $f'(a) = 0$. Then so long as $f''(x)$ is defined at the point $x = a$ either $f''(a) < 0$, $f''(a) = 0$ or $f''(a) > 0$.

Let us consider the case where $f''(a) < 0$ then from our earlier arguments this means that $f'(x)$ is decreasing at $x = a$; and since $f'(a) = 0$, it follows that $f'(x) > 0$ for values of x just less than a and $f'(x) < 0$ for values of x just greater than a. We therefore conclude that $x = a$ corresponds to a local maximum. This agrees with our observations in Figure 4.5 (b) since a function is concave down at a local maximum.

Similar arguments lead to the fact that if $f''(a) > 0$ then the stationary point $x = a$ corresponds to a local minimum. Which again agrees with the observation that a graph is concave up at a local minimum.

Summarising we have

- The function $f(x)$ has a **local maximum** at $x = a$ provided that $f'(a) = 0$ and $f''(a) < 0$.
- The function $f(x)$ has a **local minimum** at $x = a$ provided that $f'(a) = 0$ and $f''(a) > 0$

Unfortunately, if both $f'(a) = 0$ and $f''(a) = 0$ then this does not imply that there is a point of inflection at $x = a$. The following method is therefore recommended to check if we have a point of inflection

- Find the points $x = a$ where $f''(a) = 0$
- Check whether $f''(x)$ changes sign as x passes through $x = a$. If this occurs then $x = a$ is a **point of inflection**.
- If in addition $f'(a) = 0$ then $x = a$ is a **stationary point of inflection**.

In Figure 4.9 we illustrate a point of inflection which occurs when a graph of a function crosses its own tangent. Note in this case that the point is not a stationary point of inflection as $f'(a) \neq 0$.

Figure 4.9: A point of inflection where $f'(a) \neq 0$

Example
Using the second derivative investigate the nature of the stationary points of the function

$$f(x) = 4x^3 - 21x^2 + 18x + 6$$

Solution
The function concerned is shown below

The derivative of this function is

$$f'(x) = 12x^2 - 42x + 18$$

and stationary points correspond to where $f'(x) = 0$. These are the roots of the quadratic equation $12x^2 - 42x + 18 = 0$ which are given by

$$x = \frac{42 \pm \sqrt{42^2 - 4(216)}}{24} = 3,\ \frac{1}{2}$$

which correspond to the points $(0.5, 10.25)$ and $(3, -21)$. The second derivative of the function is

$$f''(x) = 24x - 42$$

At the stationary point $(0.5, 10.25)$, $f''(0.5) = -30 < 0$ and so this corresponds to a local maximum. At the stationary point $(3, -21)$, $f''(3) = 30 > 0$ and so this corresponds to a local minimum.
We note that $f''(x) = 0$ at $x = 1.75$ and that $f''(x) < 0$ for $x < 1.75$ and $f''(x) > 0$ for $x > 1.75$. Thus the point $(1.75, -5.375)$ is a point of inflection, but not a stationary point of inflection as indicated by the graph.

4.7 Summary and Further Reading

The key topics from this chapter and references to further reading in James (fourth edition), James (customised edition) and Croft and Davison (second edition) are

- Basic idea and definition. Croft and Davison [pg 694-702]. James (customised and fourth edition) [pg 540-557]
- Elementary functions and rules of differentiation. Croft and Davison [pg 703-711, 719-730]. James (customised and fourth edition) [pg 558-586]
- Parametric and implicit differentiation. Croft and Davison [pg 732-740]. James (customised and fourth edition) [pg 586-592].
- Higher derivatives. Croft and Davison [pg 712-716]. James (customised and fourth edition) [pg. 592-597].

- Optimum values. Croft and Davison [pg 755-771]. James (customised and fourth edition) [pg. 600-609].

Chapter 5

Integration

In the last chapter we looked at the technique of differentiation which forms one of the two constituents of what we call calculus. In this chapter we will look at the other constituent part of calculus, known as integration.

5.1 Basic Ideas and Definitions

To help us to understand the concept of integration let us first begin with a basic physics example. Consider an object moving along a line with constant velocity u (in ms^{-1}). The distance s (in m) travelled by the object between times t_1 and t_2 (in s) is given by

$$s = u(t_2 - t_1) \tag{5.1}$$

This is the area 'under' the graph of the velocity function between $t = t_1$ and $t = t_2$ as shown in Figure 5.1. This, of course, deals with special case where the velocity is a constant function. However, even when the velocity varies with time, the area under the velocity graph still gives the distance travelled.

Figure 5.1: Velocity–time graph for an object moving with constant velocity u. The shaded area shows the distance travelled by the object between times t_1 and t_2.

Now let us consider the velocity time graph shown in Figure 5.2 (a). We can approximate this curve by a series of horizontal lines that either lie completely below (Figure 5.2 (b)) the curve or entirely above it (Figure 5.2 (c)).

If the approximation shown in Figure 5.2 (b) is used then it would the object at any given time would always lag behind the actual position of the object. Conversely if the approximation

(a) (b) (c)

Figure 5.2: A velocity time graph with piecewise constant approximations to it

shown in Figure 5.2 (c) is adopted then the object will always be in front of where the object should be. Thus

$$\text{distance with graph (b)} < \text{distance with graph (a)} < \text{distance with graph (c)} \tag{5.2}$$

In Figures 5.2 (b) and (c) we approximated the curve by straight line segments. In this case the shaded area can be easily calculated and is equal to distance covered by the object between times t_1 and t_2, so that

$$\text{area under graph (b)} < \text{distance with graph (a)} < \text{area under graph (c)} \tag{5.3}$$

If the horizontal steps of graphs 5.2 (b) and (c) are made very small, the difference between the areas also becomes very small. In order words, the distance for graph (a) is just the area under the graph between $t = t_1$ and $t = t_2$.

This is just one of many practical problems which involves area evaluation to determine the solution. The process of summing together all parts that make up a given area is called **integration**. The area under the curve is called the **integral** of the function. For some functions we can obtain formulae for their integrals while for others we must be content with numerical approximations.

5.1.1 Definition of an integral

Formally, we define the integral of the function $f(x)$ between $x = a$ and $x = b$ to be

$$\lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{r=1}^{n} f(x_r^*)\Delta x_{r-1} \tag{5.4}$$

Let us now look at this part by part to explain what each term means. We recall that in the above we subdivided the interval and used straight line segments to approximate the area under the curve. We introduce $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ as the points of the subdivision of the interval $[a, b]$. The width of each subdivision we define as $\Delta x_{r-1} = x_r - x_{r-1}$. The maximum of these subdivisions is Δx which is formally written as

$$\Delta x = \max_{r=0,1,\cdots n-1} \Delta x_r = \max_{r} \Delta x_r \tag{5.5}$$

The area of each of the rectangles is simply given by the product $f(x_r^*)\Delta x_{r-1}$ where $x_{r-1} \leq x_r^* \leq x_r$. If we choose $x_r^* = x_{r-1}$ this would be the same as the situation illustrated in Figure 5.2 (b), or

Figure 5.3: Illustration of the formal definition of an integral

if we choose $x_r^* = x_r$ then this would be as illustrated in Figure 5.2 (c). In the formal definition, a point between these two extremes is chosen to evaluate the function, as shown in Figure 5.3.

For a given number of subdivisions n, the summation

$$\sum_{r=1}^{n} f(x_r^*)\Delta x_{r-1} \tag{5.6}$$

approximates the area under the curve. The limit of this summation as $\Delta x \to 0$ and $n \to \infty$ just means that we make size of the subdivisions smaller and smaller while at the same time increasing the number of subdivisions so that they always completely cover the interval $[a, b]$, thus leading to the exact result for the integral.

Fortunately, a concise notation for an integral has been introduced

$$\int_a^b f(x)\mathrm{d}x \tag{5.7}$$

where the integration symbol $\int$ is like an elongated S standing for summation. The $\mathrm{d}x$ is called the **differential** of x, and a and b are called the **limits of integration**. The function which is being integrated is called the **integrand**.

This is equal to the formal definition of the integral, so that

$$\int_a^b f(x)\mathrm{d}x = \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \sum_{r=1}^{n} f(x_r^*)\Delta x_{r-1} \tag{5.8}$$

It follows from the definition of an integral, that if the graph of $f(x)$ is below the x axis then the summation involves the produce of negative values of $f(x_r^*)$ with positive widths Δx_{r-1}, so that areas below the x axis must be interpreted as being negative.

5.2 Definite and Indefinite Integrals

In the previous section we have seen that the area under the graph $y = f(x)$ between $x = a$ and $x = b$ is given by the integral

$$\int_a^b f(x)\mathrm{d}x \tag{5.9}$$

This area will depend on a and b as well as the function $f(x)$. We therefore say that the integral of a function $f(x)$ may be regarded as function of a and b. This type of integral is called a **definite integral** as a and b are fixed. The result of a definite integral is always a number.

If we replace b by the variable x, we obtain a function F that is equal to the area under the graph between a and x, as shown in Figure 5.4. In this case we have

$$F(x) = \int_a^x f(t)\mathrm{d}t \tag{5.10}$$

we call this type of integral an **indefinite integral**. Note that the result of this integral is a function and not a value. In the indefinite integral, a dummy variable has to be used as the integration (here t), this must be chosen to different from the variable x on which the function depends.

(a) (b)

Figure 5.4: Illustration of an indefinite integral, showing (a) the integrand and (b) the function resulting from the integration

Now let another indefinite integral which has a different lower limit of integration c which is such that $a < c < x$. A different function is then obtained

$$G(x) = \int_c^x f(t)\mathrm{d}t \tag{5.11}$$

Inspecting the functions F and G graphically indicates that they only differ by a constant, as can be seen in Figure 5.5

$$F(x) - G(x) = \int_a^x f(t)\mathrm{d}t - \int_c^x f(t)\mathrm{d}t = \int_a^c f(t)\mathrm{d}t \tag{5.12}$$

which is an definite integral having constant value equal to the area under the graph between a and c.

When no lower limit of integration is specified we denote the indefinite integral as

$$\int f(x)\mathrm{d}x \qquad \text{or} \qquad \int^x f(x)\mathrm{d}x \tag{5.13}$$

and always include an **arbitrary constant of integration**. Thus

$$\int f(x)\mathrm{d}x = H(x) + c \tag{5.14}$$

where c is the arbitrary constant of integration. For indefinite integrals with a given lower limit of integration this constant can be determined.

Figure 5.5: Illustration that the difference of two indefinite integrals is a definite integral

In turns out that we can express definite integrals in turns of indefinite integrals. If we set $H(x) + c = \int f(x)\mathrm{d}x$ then

$$\int_a^b f(x)\mathrm{d}x = H(b) - H(a) \tag{5.15}$$

which is often written as

$$\int_a^b f(x)\mathrm{d}x = [H(x)]_a^b \tag{5.16}$$

a notation that was first introduced by Fourier.

We saw earlier that functions are only differentiable at points where the graph has a unique tangent. Functions which are not differentiable are often integrable, with the corresponding indefinite integrals being functions having a **smooth graphs**. For this this reason, engineers often call integration a smoothing process.

5.2.1 The fundamental theorem of calculus

Let us consider some graphs of some simple functions which we can compute the area under the curve by hand. In Figure 5.6 (a) we show the graph of the function $f(x) = x$, the definite integral of this function between a and b is clearly

$$\int_b^a 1\,\mathrm{d}x = b - a \tag{5.17}$$

which means that its indefinite integral is

$$\int 1\,\mathrm{d}x = x + \text{constant} \tag{5.18}$$

Similarly the area under the graph of the function $f(x) = x$, shown in Figure 5.6 (b), is easily computed

$$\int_a^b x\,\mathrm{d}x = \frac{1}{2}(b^2 - a^2) \tag{5.19}$$

so that the indefinite integral is given by

$$\int x\,\mathrm{d}x = \frac{1}{2}x^2 + \text{constant} \tag{5.20}$$

Comparable results for differentiation are

$$\frac{\mathrm{d}}{\mathrm{d}x}(k) = 0 \qquad k \text{ constant} \tag{5.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1 \tag{5.22}$$

(a) (b)

Figure 5.6: Shaded areas representing the definite integrals of the simple functions (a), $f(x) = 1$ and (b), $f(x) = x$

and we recall the sum rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + k] = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)] + \frac{\mathrm{d}}{\mathrm{d}x}(k) = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)] \tag{5.23}$$

If we differentiate our two earlier indefinite integrals, (5.18) and (5.20), we find that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int 1 \,\mathrm{d}x\right) = \frac{\mathrm{d}}{\mathrm{d}x}(x + \text{constant}) = 1 \tag{5.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int x \,\mathrm{d}x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}x^2 + \text{constant}\right) = x \tag{5.25}$$

This suggests a more general result that *the process of differentiation is the reverse of that of integration*. The general result is called the **Fundamental Theorem of Integral and Differential Calculus** and is stated below

The indefinite integral $F(x)$ of a continuous function $f(x)$ always possesses a derivative $F'(x)$ and moreover $F'(x) = f(x)$

5.3 Basic Techniques of Integration

When the fundamental theorem of calculus (in conjunction with the results below) is applied to the standard derivatives given in Table 4.2 we obtain the standard integrals given in Table 5.1

To allow us to integrate a wider range of integrals, the following rules maybe applied

- **Rule 1 (scalar–multiplication rule)**
  If k is a constant then

$$\int kf(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x \tag{5.26}$$

- **Rule 2 (sum rule)**

$$\int [f(x) \pm g(x)]\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x \tag{5.27}$$

- **Rule 3 (linear composite rule)**
  If a and b are constants and $F'(x) = f(x)$ then

$$\int f(ax + b)\,\mathrm{d}x = \frac{1}{a}F(ax + b) + \text{constant} \tag{5.28}$$

$f(x)$	$\int f(x)\,\mathrm{d}x$ Here c is a constant of integration
$x^n \quad (n \neq -1)$	$\frac{1}{n+1}x^{n+1} + c$
$\frac{1}{x}$	$\left.\begin{array}{ll} \ln x + c & (x > 0) \\ \ln(-x) + c & (x < 0) \end{array}\right\} = \ln\lvert x\rvert + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\tan x$	$\ln\lvert\sec x\rvert + c$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + c$
$\operatorname{cosec} x$	$\ln\left\lvert\tan\frac{x}{2}\right\rvert + c$
$\cot x$	$\ln\lvert\sin x\rvert + c$
e^x	$e^x + c$
$\ln x$	$x\ln x - x + c$
$\sin^{-1} x$	$x\sin^{-1} x + \sqrt{1-x^2} + c$
$\cos^{-1} x$	$x\cos^{-1} x - \sqrt{1-x^2} + c$
$\tan^{-1} x$	$\frac{1}{2}[2x\tan^{-1} x - \ln(1+x^2)] + c$
$\sinh x$	$\cosh x + c$
$\cosh x$	$\sinh x + c$
$\tanh x$	$\ln(\cosh x) + c$

Table 5.1: Some standard functions and their integrals

- **Rule 4 (inverse–function rule)**
  If $y = f^{-1}(x)$, so that $x = f(y)$, then

$$\int f^{-1}(x)\,\mathrm{d}x = xy - \int f(y)\,\mathrm{d}y \tag{5.29}$$

Below we consider some examples of the approach

Example
Find the indefinite of $2x^2$
Solution
We use the scalar multiplication rule

$$\int 2x^2\mathrm{d}x = 2\int x^2\mathrm{d}x = \frac{2}{3}x^3 + c$$

Example
Determine the indefinite integral of $6x^4 + 4x - \frac{3}{x}$
Solution
Using the sum rule

$$\begin{aligned} \int 6x^4 + 4x - \frac{3}{x}\,\mathrm{d}x &= \int 6x^4\,\mathrm{d}x + \int 4x\,\mathrm{d}x - \int \frac{3}{x}\,\mathrm{d}x \\ &= \frac{6}{5}x^5 + \frac{4}{2}x^2 - 3\ln\lvert x\rvert + c \\ &= \frac{6}{5}x^5 + 2x^2 - 3\ln\lvert x\rvert + c \end{aligned}$$

Example
Determine the indefinite integral of $\sqrt{5x+2}$
Solution
Using the linear composite rule

$$\begin{aligned}\int \sqrt{5x+2}\,\mathrm{d}x &= \frac{1}{5}\left[\frac{2}{3}(5x+2)^{3/2}\right]+c \\ &= \frac{2}{15}(5x+2)^{3/2}+c\end{aligned}$$

Example
Determine the indefinite integral of $\ln x$
Solution
If $y = \ln x$ then $x = e^y$ and using the inverse function rule

$$\begin{aligned}\int \ln x\,\mathrm{d}x &= xy - \int e^y \mathrm{d}y \\ &= xy - e^y + c \\ &= x\ln x - x + c\end{aligned}$$

Example
Determine the indefinite integral of $\sin^{-1} x$
Solution
If $y = \sin^{-1} x$ then $x = \sin y$ and using the inverse function rule

$$\begin{aligned}\int \sin^{-1} x\,\mathrm{d}x &= xy - \int \sin y \mathrm{d}y \\ &= xy + \cos y + c \\ &= x\sin^{-1} x + \sqrt{1-x^2} + c\end{aligned}$$

where the last result follows from the identity $\sin^2 y + \cos^2 y = 1$.

5.4 Integrals Involving Partial Fractions

When faced with integrals involving partial fractions, we first expand the partial faction and then integrate each term separately, as illustrated in the next example.

Example
Using partial fractions, evaluate the indefinite integral of $\frac{6}{x^2-2x-8}$
Solution
We first expand $\frac{6}{x^2-2x-8}$ in terms of partial fractions

$$\frac{6}{x^2-2x-8} = \frac{6}{(x+2)(x-4)} = \frac{-1}{(x+2)} + \frac{1}{x-4}$$

We now evaluate the integral

$$\begin{aligned}\int \frac{6}{x^2-2x-8}\,\mathrm{d}x &= \int \frac{-1}{(x+2)}\,\mathrm{d}x + \int \frac{1}{x-4}\,\mathrm{d}x \\ &= -\ln|x+2| + \ln|x-4| + c \\ &= \ln\left|\frac{x-4}{x+2}\right| + c\end{aligned}$$

5.5 Additional Rules for Definite Integrals

When considering definite integrals, the following rules also apply

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x \tag{5.30}$$

and

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \qquad (a \le c \le b) \tag{5.31}$$

which is us to break the interval $[a, b]$ in to convenient chunks to perform the integration.

5.6 Integration by Parts

We can use the product rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = \frac{\mathrm{d}u}{\mathrm{d}x}v + \frac{\mathrm{d}v}{\mathrm{d}x}u \tag{5.32}$$

to allow us to integrate yet more functions. We rearrange it in the form

$$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - v\frac{\mathrm{d}u}{\mathrm{d}x} \tag{5.33}$$

and then integrate it to obtain

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \tag{5.34}$$

We can use this result to integrate the product of two functions using a method known as **integration by parts**: We choose one term to be u and the other to be $\frac{\mathrm{d}v}{\mathrm{d}x}$. Then, we calculate $\frac{\mathrm{d}u}{\mathrm{d}x}$ and v and substitute the appropriate terms in to the right hand side of equation (5.34).

Example
Find the indefinite integral of $x^2 \cos x$
Solution
We first choose $u = x^2$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$ so that $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and $v = \sin x$. Then

$$\begin{aligned}\int x^2 \cos x \,\mathrm{d}x &= x^2 \sin x - \int (\sin x)(2x)\mathrm{d}x \\ &= x^2 \sin x - \int 2x \sin x \mathrm{d}x\end{aligned}$$

The same technique is now applied to the last integral. Here we choose $u = 2x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$ giving $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$ and $v = -\cos x$. Applying the integration by parts formula for the second time gives

$$\begin{aligned}\int x^2 \cos x \,\mathrm{d}x &= x^2 \sin x - \left[(2x)(-\cos x) - \int (-\cos x)(2)\,\mathrm{d}x\right] \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + c\end{aligned}$$

5.7 Integration by Substitution

The last integration method we shall consider is called integration by substitution. The procedure is obtained by first considering the composite rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x) \tag{5.35}$$

By integrating, we reverse the differentiation process

$$\int f'(g(x))g'(x)\,\mathrm{d}x = f(g(x)) + c \tag{5.36}$$

where c is a constant. The key step is to identify the function $g(x)$, note that this will not be unique and different choice of $g(x)$ may differ by a constant. To simplify the process we set $t = g(x)$ so that the integral becomes

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(t)\frac{\mathrm{d}t}{\mathrm{d}x}\mathrm{d}x = \int f'(t)\,\mathrm{d}t = f(t) + c \tag{5.37}$$

$$= f(g(x))) + c \qquad \text{on back substitution} \tag{5.38}$$

This technique is called **integration by substitution**.

Example
Find the indefinite integral of $2x\sqrt{x^2+3}$
Solution
A suitable substitution is $t = x^2 + 3$, with $\frac{\mathrm{d}t}{\mathrm{d}x} = 2x$

$$\begin{aligned}\int 2x\sqrt{x^2+3}\,\mathrm{d}x &= \int \sqrt{t}\frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \int \sqrt{t}\,\mathrm{d}t \\ &= \frac{2}{3}t^{3/2} + c \\ &= \frac{2}{3}(x^2+3)^{3/2} + c\end{aligned}$$

Example
Find the integral of $\int_0^1 \frac{x+1}{x^2+2x+2}\mathrm{d}x$
Solution
A suitable substitution is $t = x^2 + 2x + 2$ so that $\frac{\mathrm{d}t}{\mathrm{d}x} = 2x + 2$. However, when we make this substitution the limits of integration will change so that when $x = 0$, $t = 2$ and when $x = 1$, $t = 5$

$$\begin{aligned}\int_0^1 \frac{x+1}{x^2+2x+2}\mathrm{d}x &= \frac{1}{2}\int_0^1 \frac{2x+2}{x^2+2x+2}\mathrm{d}x = \int_0^1 \frac{1}{2t}\frac{\mathrm{d}t}{\mathrm{d}x}\mathrm{d}x = \frac{1}{2}\int_2^5 \frac{1}{t}\mathrm{d}t \\ &= \left[\frac{1}{2}\ln|t|\right]_2^5 \\ &= \frac{1}{2}(\ln 5 - \ln 2) \\ &= \frac{1}{2}\ln 2.5 = 0.458\ (3dp)\end{aligned}$$

The last two examples were forms of two common cases. Integrals of the form

$$\int \frac{g'(x)}{g(x)}\mathrm{d}x$$

where $g'(x)$ is the derivative with respect to x of some function $g(x)$ may be determined by making the substitution $u = g(x)$. In this case $\frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$ and

$$\int \frac{g'(x)}{g(x)}\mathrm{d}x = \int \frac{\mathrm{d}u}{u} = \ln|u| + c = \ln|g(x)| + c$$

Example
Find the integral of $\int \tan ax \mathrm{d}x$
Solution
We write

$$\int \tan ax \mathrm{d}x = \int \frac{\sin ax}{\cos ax}\mathrm{d}x = -\frac{1}{a}\int \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\cos ax)}{\cos ax} = -\frac{1}{a}\ln|\cos ax| + c$$

Another common type of integral where a substitution can be used to great effect is

$$\int g'(x)(g(x))^n \mathrm{d}x$$

here $g'(x)$ is again the derivative of $g(x)$ and $n \neq -1$. If we make the substitution $u = g(x)$ then $\frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$ so that

$$\int g'(x)(g(x))^n \mathrm{d}x = \int u^n \mathrm{d}u = \frac{u^{n+1}}{n+1} + c = \frac{(g(x))^{n+1}}{n+1} + c$$

If the Integrand contains	Try	
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$\frac{dx}{d\theta} = a\cos\theta$
	or $x = a\tanh u$	$\frac{dx}{du} = a\,\mathrm{sech}^2\, u$
$\sqrt{a^2 + x^2}$	$x = a\sinh u$	$\frac{dx}{du} = a\cosh u$
	or $x = a\tan\theta$	$\frac{dx}{d\theta} = a\sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\cosh u$	$\frac{dx}{du} = a\sinh u$
	or $x = a\sec\theta$	$\frac{dx}{d\theta} = a\sec\theta\tan\theta$
Circular Functions	$s = \sin x$	$\frac{ds}{dx} = \cos x$
	or $c = \cos x$	$\frac{dc}{dx} = -\sin x$
	or $t = \tan\frac{1}{2}x$	
	$\left(\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \frac{dx}{dt} = \frac{2}{1+t^2}\right)$	
Hyperbolic Functions	$u = e^x$	$\frac{du}{dx} = e^x$
	or $s = \sinh x$	$\frac{ds}{dx} = \cosh x$
	or $c = \cosh x$	$\frac{dc}{dx} = \sinh x$
	or $t = \tanh x$	$\frac{dt}{dx} = \frac{1}{2}\,\mathrm{sech}^2\,\frac{1}{2}x$

Table 5.2: List of possible substitution for more complicated integration by substitution problems

Example
Find the integral of $\int \frac{(\ln x)^2}{x} dx$
Solution
We write

$$\int \frac{(\ln x)^2}{x} dx = \int \frac{1}{x} \cdot (\ln x)^2 dx = \int \frac{d}{dx}(\ln x)(\ln x)^2 dx = \frac{(\ln x)^3}{3} + c$$

For more complicated cases choosing to correct substitution is not always an easy task. To aid the choice of substitution, Table 5.2 gives a list of some possible choices for more complex integrals.

Example
Find the integral of $\int \sqrt{1 - x^2} dx$
Solution
The integrand contains $\sqrt{a^2 - x^2}$ with $a = 1$ so we try the substitution $x = \sin\theta$, $dx/d\theta = \cos\theta$

$$\int \sqrt{1 - x^2} dx = \int \sqrt{1 - \sin^2\theta}\cos\theta d\theta = \int \cos^2\theta d\theta$$

Although this simpler than the original integral it is still not immediately integrable. However, by using the identity $\cos 2\theta = 2\cos^2\theta - 1$ gives

$$\int \sqrt{1 - x^2} dx = \int \frac{1}{2}(\cos 2\theta + 1) d\theta = \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c = \frac{1}{4}\sin(2\sin^{-1} x) + \frac{1}{2}\sin^{-1} x + c$$

Or by using the identity $\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1 - \sin^2\theta}$ we can write this in the alternative form

$$\int \sqrt{1 - x^2} dx = \frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1} x + c$$

5.8 Integration of More Complicated Trigonometric Functions

Powers and product of sines and cosines also frequently occur in integrands. Using the techniques we have learnt we can also evaluate these integrals.

Example
Find the integral of $\sin 2x \cos 2x$
Solution
Make the substitution $u = \sin 2x$, $\mathrm{d}u/\mathrm{d}x = 2\cos 2x$ and so

$$\int \sin 2x \cos 2x \mathrm{d}x = \frac{1}{2}\int 2\sin 2x \cos 2x \mathrm{d}x = \frac{1}{2}\int u \mathrm{d}u = \frac{1}{4}u^2 + c$$

Back substituting gives

$$\int \sin 2x \cos 2x \mathrm{d}x = \frac{1}{4}\sin^2 2x + c$$

Example
Find the integral of $\sin^3 x$
Solution
First write $\sin^3 x = \sin x \sin^2 x$ and then use the identity $\sin^2 x = 1 - \cos^2 x$

$$\int \sin^3 x \mathrm{d}x = \int \sin x \sin^2 x \mathrm{d}x = \int \sin x \mathrm{d}x - \int \sin x \cos^2 \mathrm{d}x = -\cos x - \int \sin x \cos^2 x \mathrm{d}x$$

For the remaining integral use the substitution $c = \cos x$, $\mathrm{d}c/\mathrm{d}x = -\sin x$ so

$$\int \sin^3 x \mathrm{d}x = -\cos x + \int c^2 \mathrm{d}c = -\cos x + \frac{c^3}{3} + c = -\cos x + \frac{\cos^3 x}{3} + c$$

Example
Find the integral of $\sin 4x \cos 2x$
Solution
Apply the trigonometric identity $2\sin A \cos B = \sin(A+B) + \sin(A-B)$, which is obtained by adding $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and $\sin(A-B) = \sin A \cos B - \cos A \sin B$. Set $A = 4x$ and $B = 2x$ to give

$$\int \sin 4x \cos 2x \mathrm{d}x = \frac{1}{2}\int (\sin(2x+4x) + \sin(4x-2x))\,\mathrm{d}x = \frac{1}{2}\int (\sin 6x + \sin 2x)\,\mathrm{d}x$$

which is integrable and leads to

$$\int \sin 4x \cos 2x \mathrm{d}x = -\frac{1}{12}\cos 6x - \frac{1}{4}\cos 2x + c$$

5.9 Summary and Further Reading

The key topics from this chapter and references to further reading in James (fourth edition), James (customised and fourth edition) and Croft and Davison (second edition) are

- Basic ideas and definitions. Croft and Davison [pg 774-775]. James (customised and fourth edition) [pg 613-620]
- Definite and indefinite integrals. Croft and Davison [pg 775, 786-792]. James (customised and fourth edition) [pg 620-622].
- Fundamental Theorem of calculus. Croft and Davison [pg 774-780]. James (customised and fourth edition) [pg 623-625].
- Rules of integration. Croft and Davison [pg 781-785]. James (customised and fourth edition) [pg 625-630].
- Partial fractions. Croft and Davison [pg 833-835]. James (customised and fourth edition) [pg 631-632].
- Integration by parts. Croft and Davison [pg 815-821]. James [pg 637-640].
- Integration by substitution. Croft and Davison [pg 822-832]. James (customised and fourth edition) [pg 640-646].

Chapter 6

Linear Algebra

In this chapter we will introduce the concept of Linear Algebra. No doubt you have come across set of two or three simultaneous equations. We shall extend the idea of simultaneous equations in to the wider field of linear algebra. In doing so, we shall introduce a new notation for representing the equations, called matrices and a solution process called Gauss elimination.

When exploring solutions to sets of simultaneous equations, we need to know what solutions we should expect, a single set of values or no solution, infinitely many solutions and this is where are discussions begin.

6.1 Simultaneous Equations

Let us start with an example.

Example

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned}$$

Solution

This is an example for linear equation system with two equations and two unknowns. We wish to find x_1 and x_2 so that both equations are fulfilled. The values $x_1 = 1$ and $x_2 = 2$, which are found by substituting one equation in to another, satisfy both equations. They represent the solution of the linear equation system.

Note that in general a linear equation system may have m equations and n unknowns. Lets look at some more examples.

Example

$$\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 + 2x_2 &= 5 \end{aligned}$$

Here $m = 2$ and $n = 2$.

Solution

This is linear equation system has *no* solution. If one multiplies the first equation by 2 one obtains $2x_1 + 2x_2 = 8$ which disagrees with the second equation.

Example

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$$

Here $m = 2$ and $n = 3$.
Solution
One possible solution is $x_1 = 2$, $x_2 = 0$ and $x_3 = 0$ another is $x_1 = 2$, $x_2 = 1$ and $x_3 = 1$. In general there are infinitely many solutions, namely $x_1 = 2$, $x_2 = \alpha$ and $x_3 = \alpha$ where α is any real number.

Example

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 &= 4 \end{aligned}$$

Here $m = 3$ and $n = 2$.
Solution
This is linear equation system has *no* solution. Through addition of the first two equations, one has $2x_1 = 3$ which disagrees with the last equation.

The set of all solutions of a linear equation system is called the **solution set** of the linear equation system.

6.2 Gauss Elimination

The aim of this section is to describe the development of an efficient strategy for determining the solution set of a linear equation system. By efficient, we mean that with the fewest possible calculations. The procedure that we describe is called Gauss elimination. The idea of Gauss elimination is to transform the linear system so that it is easier to solve. Note that the transformation is performed so that the solution set is not changed during the process.

The following two operations transform a linear equation system in to an equivalent system.

Exchanging equations

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned} \quad \text{is equivalent to} \quad \begin{aligned} 2x_1 + 3x_2 &= 8 \\ x_1 + 2x_2 &= 5 \end{aligned}$$

It is clear that both equation systems posses the same solution set.

Addition of factored equation to another equation

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned} \quad \text{is equivalent to} \quad \begin{aligned} x_1 + 2x_2 &= 5 \\ -x_2 &= -2 \end{aligned}$$

Here, the first equation remains the same. To obtain the revised second equation, we must multiplied the first equation by 2 and then subtracted it from the second equation in the original system. The solution set of the left equation system is the same as solution set of the equation system on the right.

A linear equation with n equations and n unknowns is easier to solve when it is has a triangular form. In this case, one can easily obtain the solution set by back substitution.

Example

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

Here $m = n = 3$.

Solution

From the third equation we have $x_3 = 2$. Substituting this in to the second equation gives $x_2 = 4$ and finally using both values in the first equation gives $x_1 = -3$.

Lets now attempt to solve a linear equation system using these ideas.

Example

$$\begin{aligned} 2x_2 + 2x_3 &= 1 \\ 2x_1 + 4x_2 + 5x_3 &= 9 \\ x_1 - x_2 + 2x_3 &= 3 \end{aligned}$$

Here $m = n = 3$.

Solution

First we exchange the first and second equations

$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 &= 9 \\ 2x_2 + 2x_3 &= 1 \\ x_1 - x_2 + 2x_3 &= 3 \end{aligned}$$

Next we multiply the first equation by $\frac{1}{2}$ and subtract it from the third equation

$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 &= 9 \\ 2x_2 + 2x_3 &= 1 \\ -3x_2 - \frac{1}{2}x_3 &= -\frac{3}{2} \end{aligned}$$

Finally we multiply the second equation by $\frac{3}{2}$ and add it from the third equation, giving

$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 &= 9 \\ 2x_2 + 2x_3 &= 1 \\ \frac{5}{2}x_3 &= 0 \end{aligned}$$

Then by back substitution we find the solution $x_1 = \frac{7}{2}$, $x_2 = \frac{1}{2}$ and $x_3 = 0$. It therefore follows that this is the only solution to the linear system, i.e. the linear set contains only this solution.

What we have just performed is the Gauss elimination algorithm, to explain it in more details let us consider a system where $m = n = 3$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{13}x_3 &= b_3 \end{aligned}$$

here the values a_{ij}, $i = 1, \cdots, 3$ and $j = 1, \cdots, 3$ and b_i, $i = 1, \cdots, 3$ are known real numbers. The values a_{ij} are the **coefficients** of the unknown x_j in the ith equation of the system. The number b_i is the right hand side of the ith equation.

6.2.1 Schematic representation

To aid the application of the Gauss elimination algorithm, we express the system in the form

x_1	x_2	x_3	1
a_{11}	a_{12}	a_{13}	b_1
a_{21}	a_{22}	a_{23}	b_2
a_{31}	a_{32}	a_{33}	b_3

The coefficients are written in the main part of the schematic, in the header–row stand the respective unknowns x_1, x_2 and x_3. The right–hand side is labelled the 1–column. The schematic is just another way of writing the linear system. When we wish to perform operations on the equations, we just perform the analogue on the schematic.

Step 1

Assumption: One of a_{i1}, $i = 1, \cdots, 3$ is not zero, in other words at least one values in the first column isn't zero.

a) If $a_{11} = 0$ then swap the first row with a row whose first element isn't zero. Next we re-name the coefficients according to the position in which they are now lying in. Continue to case b).

b) If $a_{11} \neq 0$ we create a new equivalent scheme in which the coefficients in the second row are given by their original value minus the value in the first row factored by a_{21}/a_{11}. Similarly, the values in the third row are given by their original value minus the value in the first row factored by a_{31}/a_{11}.

As a result of this operation the first number in the second and third row, will now be zero. To highlight the fact that the values in the second and third column have changed we label them with the superscript (2). We have now eliminated the unknown x_1 from the second and third equations:

x_1	x_2	x_3	1
a_{11}	a_{12}	a_{13}	b_1
0	$a_{22}^{(2)}$	$a_{23}^{(2)}$	$b_2^{(2)}$
0	$a_{32}^{(2)}$	$a_{33}^{(2)}$	$b_3^{(2)}$

Step 2

Assumption:One of $a_{i2}^{(2)}$, $i = 2, 3$ is non zero.

a) If $a_{22}^{(2)} = 0$ we swap rows and then proceed to b).

b) If $a_{22}^{(2)} \neq 0$ we create a new equivalent system whose entries in the third row are given by their original values minus the values in the second row factored by $a_{32}^{(2)}/a_{22}^{(2)}$.

We have now eliminated the unknown x_2 from the third equation, resulting in a scheme with triangular form

x_1	x_2	x_3	1
a_{11}	a_{12}	a_{13}	b_1
0	$a_{22}^{(2)}$	$a_{23}^{(2)}$	$b_2^{(2)}$
0	0	$a_{33}^{(3)}$	$b_3^{(3)}$

Step3

The equivalent system is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 &= b_2^{(2)} \\ a_{33}^{(3)}x_3 &= b_3^{(3)} \end{aligned}$$

The solution set can be found by back substitution.

6.2.2 Algorithm for $m = n$

What we have just performed can be expressed in terms of an algorithm. In the algorithm we use the construction **For** $j = 1, \cdots, n$ which means that all the lines which follow this statement are evaluated with $j = 1$ until the statement **End** is reached. At this point the lines are then executed again, in turn, but this time with $j = 2$ until **End** is reached. The process is repeated until $j = n$. This enables us to write the Gauss elimination algorithm in a concise way and will turn out to be useful for The algorithm for the case where we have n equations and n unknowns is as follows:

Perform the elimination process:

For $j = 1, \cdots, n-1$:

Determine the row index $p \in j, \cdots, n$ for which $a_{pj}^{(j)} \neq 0$

If $p \neq j$ exchange rows and renumber coefficients according to their new locations.

For $k = j+1, \cdots, n$:

Compute $l_{kj} = a_{kj}^{(j)}/a_{jj}^{(j)}$

For $p = j, \cdots, n$:

Set $a_{kp}^{(j+1)} = a_{kp}^{(j)} - l_{kj}a_{jp}^{(j)}$

End

Set $b_k^{(j+1)} = b_k^{(j)} - l_{kj}b_j^{(j)}$

End

End

Determine the solution by back substitution.

Set $x_n = b_n/a_{nn}$

For $j = n-1, \cdots, n$

Set $v = b_j$

For $k = j+1, \cdots, n$

Calculate $v = v - a_{jk}x_k$

End

Set $x_j = v/a_{jj}$

End

Example

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &= 5 \\ 2x_1 + x_2 + x_3 + x_4 &= 3 \\ x_1 + 2x_2 + x_3 &= 4 \\ x_2 + x_3 + 2x_4 &= 0 \end{aligned}$$

Solution

First we write the equation in schematic representation

x_1	x_2	x_3	x_4	1
1	2	3	1	5
2	1	1	1	3
1	2	1	0	4
0	1	1	2	0

We now proceed with the Gauss elimination algorithm:

x_1	x_2	x_3	x_4	1
1	2	3	1	5
0	-3	-5	-1	-7
0	0	-2	-1	-1
0	1	1	2	0

x_1	x_2	x_3	x_4	1
1	2	3	1	5
0	-3	-5	-1	-7
0	0	-2	-1	-1
0	0	$-\frac{2}{3}$	$\frac{5}{3}$	$-\frac{7}{3}$

x_1	x_2	x_3	x_4	1
1	2	3	1	5
0	-3	-5	-1	-7
0	0	-2	-1	-1
0	0	0	$\frac{6}{3}$	$-\frac{6}{3}$

We then determine the solution through back substitution giving $x_4 = -1$, $x_3 = 1$, $x_2 = 1$ and $x_1 = 1$

What happens when we apply the Gauss elimination algorithm to a system that has no solution? Lets consider the following example

Example

$$\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 + 2x_2 &= 5 \end{aligned}$$

Solution
First we write the equation in schematic representation

x_1	x_2	1
1	1	4
2	2	5

Then we proceed with the Gauss elimination algorithm, giving

x_1	x_2	1
1	1	4
0	0	-3

Clearly $0x_1 + 0x_2 \neq -3$ so the linear system has no solution.

Let us now look at an example in which the right hand side vector contains a unknown parameter.

Example

$$\begin{aligned} 2x_1 - x_2 + 3x_3 - x_4 + x_5 &= -2 \\ 2x_1 - x_2 + 3x_3 - x_5 &= -3 \\ -4x_1 + 2x_2 - 4x_3 + 5x_4 - 5x_5 &= 3 \\ -2x_3 + 2x_4 - 7x_5 &= -5 + s \\ -2x_1 + x_2 - x_3 + 4x_5 &= 5 \end{aligned}$$

Solution
First we write the equation in schematic representation

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
2	-1	3	0	-1	-3
-4	2	-4	5	-5	3
0	0	-2	2	-7	-5+s
-2	1	-1	0	4	5

then we proceed with the Gauss elimination algorithm:

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
0	0	0	1	-2	-1
0	0	2	3	-3	-1
0	0	-2	2	-7	-5+s
0	0	2	-1	5	3

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
0	0	2	3	-3	-1
0	0	0	1	-2	-1
0	0	-2	2	-7	-5+s
0	0	2	-1	5	3

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
0	0	2	3	-3	-1
0	0	0	1	-2	-1
0	0	0	5	-10	-6+s
0	0	0	-4	8	4

x_1	x_2	x_3	x_4	x_5	1
2	-1	3	-1	1	-2
0	0	2	3	-3	-1
0	0	0	1	-2	-1
0	0	0	0	0	-1+s
0	0	0	0	0	0

We have the following consequences. For $s \neq 1$ we have no solution to the linear system. For $s = 1$, through back substitution we find

$$\begin{aligned} x_4 &= -1 + 2x_5 \\ x_3 &= \frac{1}{2}(-1 + 3x_5 - 3x_4) = 1 - \frac{3}{2}x_5 \\ x_1 &= \frac{1}{2}(-2 - x_5 + x_4 - 3x_3 + x_2) = -3 + \frac{11}{4}x_5 + \frac{1}{2}x_2 \end{aligned}$$

The solution set consists of two free parameters x_5 and x_2, this means that for $s = 1$ we have infinitely many solutions to the linear system, giving

$$\begin{aligned} x_1 &= = -3 + \frac{11}{4}\beta + \frac{1}{2}\alpha \\ x_2 &= \alpha \\ x_3 &= 1 - \frac{3}{2}\beta \\ x_4 &= -1 + 2\beta \\ x_5 &= \beta \end{aligned}$$

where α and β are any real numbers.

6.2.3 Linear systems of equations with multiple right hand sides

To illustrate how the Gauss elimination method may be used with problems with multiple right hand side vectors, consider the following example

Example

$$\begin{aligned} 2x_2 + 2x_3 &= b_1 \\ 2x_1 + 4x_2 + 5x_3 &= b_2 \\ x_1 - x_2 + 2x_3 &= b_3 \end{aligned} \qquad a) \begin{aligned} b_1 &= 1 \\ b_2 &= 9 \\ b_3 &= 3 \end{aligned} \qquad b) \begin{aligned} b_1 &= 2 \\ b_2 &= 13 \\ b_3 &= 1 \end{aligned} \qquad c) \begin{aligned} b_1 &= 5 \\ b_2 &= -4 \\ b_3 &= 2 \end{aligned}$$

Solution

x_1	x_2	x_3	1_a	1_b	1_c
0	2	2	1	2	5
2	4	5	9	13	-4
1	-1	2	3	1	2

x_1	x_2	x_3	1_a	1_b	1_c
1	-1	2	3	1	2
2	4	5	9	13	-4
0	2	2	1	2	5

x_1	x_2	x_3	1_a	1_b	1_c
1	-1	2	3	1	2
0	6	1	3	11	-8
0	2	2	1	2	5

x_1	x_2	x_3	1_a	1_b	1_c
1	-1	2	3	1	2
0	6	1	3	11	-8
0	0	$\frac{5}{3}$	0	$-\frac{5}{3}$	$\frac{23}{3}$

By backsubstition we obtain the solutions a) $x_1 = \frac{7}{2}$, $x_2 = \frac{1}{2}$, $x_3 = 0$, b) $x_1 = 5$, $x_2 = 2$, $x_3 = -1$ and c) $x_1 = -\frac{279}{30}$, $x_2 = -\frac{63}{30}$, $x_3 = \frac{23}{5}$.

6.2.4 The algorithm for m equations and n unknowns

In the general case the algorithm is as follows
Set $i = 1, j = 1$
* **If** $i > m$ **or** $j > n$ **then**

Goto back substitution stage

Else

If possible determine a row index $p \in \{i, \cdots, m\}$ for which $a_{pj}^{(j)} \neq 0$ otherwise set $j = j+1$ and goto *.

If $p \neq i$, exchange rows p and i and renumber them accordingly

For $k = i+1, \cdots, m$:

Compute $l_{ki} = a_{kj}^{(j)}/a_{ij}^{(j)}$
For $p = j, \cdots, n$:
Set $a_{kp}^{(j+1)} = a_{kp}^{(j)} - l_{ki}a_{ip}^{(j)}$
End
Set $b_k^{(j+1)} = b_k^{(j)} - l_{ki}b_i^{(j)}$

End

Set $i = i+1$ and $j = j+1$, goto *.

End
Compute the solution set by back substitution

6.2.5 Rank of a linear system of equations

The rank of a linear equation system is defined as the number, r, of non–zero rows after performing the Gauss elimination algorithm. Using the rank, one can immediately describe the solution set of the linear system of equations: A linear equation system has at least one solution if

- $r = m$, or
- $r < m$ and $c_i = 0$, $i = r+1, \cdots, m$ where c is the right hand side vector after Gauss elimination.

Example

$$x_1 - x_2 + x_3 = 2$$
$$2x_1 + x_2 - x_3 = 4$$

Solution
First we write the equation in schematic representation

x_1	x_2	x_3	1
1	-1	1	2
2	1	-1	4

Then we proceed with the Gauss elimination algorithm, giving

x_1	x_2	x_3	1
1	-1	1	2
0	3	-3	0

The rank of this system is therefore $r = 2$ and we have at least one solution, explicitly

$$x_2 = x_3$$
$$x_1 = 2 + x_2 - x_3 = 2$$

which means that x_3 is a free parameter. Thus we have infinitely many solutions, $x_1 = 2$, $x_2 = x_3 = \alpha$ where α is any real number.

Example

$$x_1 + x_2 = 2$$
$$x_1 - x_2 = 1$$
$$x_1 = 4$$

Solution
First we write the equation in schematic representation

x_1	x_2	1
1	1	2
1	-1	1
1	0	4

Then we proceed with the Gauss elimination algorithm, giving

x_1	x_2	1
1	1	2
0	-2	-1
0	-1	2

x_1	x_2	1
1	1	2
0	-2	-1
0	0	$\frac{5}{2}$

The rank of this system is $r = 2 < m = 3$ and $c_3 \neq 0$ so that the system has no solution.

6.3 Matrices

In the last section we introduced the Gauss elimination method for solving linear equations. To simplify the way we write large systems of equations a new notation is introduced called **matrix notation**. Matrices can be used not only in connection with linear systems of equations, but also in mappings and in the solution of systems of differential equations.

6.3.1 Matrix definitions

A $m \times n$ matrix is a schematic of mn numbers ordered in to m rows and n columns. The mn numbers are called **elements** of the $m \times n$ matrix.

In these lecture notes we shall use capital letters to represent matrices. The element of a matrix A which lies on the ith row and jth column is denoted by a_{ij} or $(A)_{ij}$. We write a **matrix** as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

For example

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 2 \end{pmatrix}$$

is a 2×3 matrix. In the first row is the second element $(A)_{12} = a_{12} = 3$.

A $n \times n$ matrix has an equal number of rows and columns and is called a **square matrix**.

The two matrices A and B are said to be **equal** when they have the same number of rows and columns and when the respective elements of the two matrices are the same

$$A_{ij} = B_{ij} \qquad \text{for all } i, j$$

For example the following two matrices are equal

$$\begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 10/2 & 1 \\ 3-1 & 2^2 \end{pmatrix}$$

Below we introduce some common matrices which are used in engineering:

- A $m \times n$ matrix is called the **null matrix** (or zero matrix) if every element of the matrix is zero. For example, the matrix
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
is 2×3 null matrix.

- A square matrix U is called a **upper triangular matrix** if $(U)_{ij} = 0$ for $i > j$. For example
$$U = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

- A square matrix L is called a **lower triangular matrix** if $(L)_{ij} = 0$ for $i < j$. For example
$$L = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

- A $n \times n$ matrix is called a **diagonal matrix** if $(D)_{ij} = 0$ for $i \neq j$. The elements $(D)_{ii} = d_{ii}$ are called the **diagonal elements**. For a diagonal matrix with given diagonal elements, $d_{11}, d_{22}, \cdots, d_{nn}$ we write $D = \text{diag}(d_{11}, d_{22}, \cdots, d_{nn})$. For example

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \text{diag}(5, 2, 3)$$

- The $n \times n$ matrix $I_n = \text{diag}(1, 1, \cdots, 1)$ is called the **identity matrix**. For example

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- A further class of matrices are the 1-column or $n \times 1$ matrices. The $n \times 1$ matrix are commonly known as **column vectors**. We write column vectors using lower case letters. The elements of column vectors are called **components**. Components are only identified with a single index. For example, the 4×1 matrix

$$b = \begin{pmatrix} 2 \\ -4 \\ 7 \\ 0 \end{pmatrix}$$

is a column vector. We also have that $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ with $b_1 = 2$, $b_2 = -4$, $b_3 = 7$ and $b_4 = 0$.

6.3.2 Computations with matrices

Addition

Consider two $m \times n$ matrices A and B. To add the matrices A and B together, we add the respective elements of A and B together. Written more precisely: the $m \times n$ matrix $A + B$ with $(A)_{ij} + (B)_{ij}$ is called the sum of matrices A and B

Example

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Find $A + B$.

Solution

$$A + B = \begin{pmatrix} 3 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix}$$

Multiplication by a scaler

If a $m \times n$ matrix is multiplied by scaler number α, this means that every element of the matrix is multiplied by α. The matrix αA with $(\alpha A)_{ij} = \alpha (A)_{ij}$ is called the α multiple of the matrix A.

Multiplication of two matrices

Let A be an $m \times n$ matrix and B a $n \times p$ matrix. The $m \times p$ matrix AB, with $(AB)_{ij} = \sum_{k=1}^{n} (A)_{ik}(B)_{kj}$ is called the matrix product of matrices A and B.

Note that the matrix product AB can only be computed when the number of columns of matrix A is exactly the same as the number of rows of matrix B. An illustration of matrix multiplication is shown in Figure 6.1. In this figure, we observe how row i of matrix A is multiplied by column

Figure 6.1: Illustration of matrix multiplication

j of matrix B to obtain the element $(AB)_{ij}$ of matrix AB. Explicitly this given as

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Example

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \text{ is a } 2 \times 3 \text{ matrix and } B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & -1 & -1 & 2 \end{pmatrix} \text{ is a } 3 \times 4 \text{ matrix}$$

Find AB.

Solution

$AB = \begin{pmatrix} 4 & 5 & 2 & 1 \\ 2 & -3 & -5 & 0 \end{pmatrix}$. The two elements in the first column were computed as follows

$$(AB)_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = 3 \cdot 1 + 1 \cdot 1 + 0 \cdot 2 = 4$$
$$(AB)_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = 2 \cdot 1 + (-2) \cdot 1 + 1 \cdot 2 = 2$$

Rules

When computing with matrices, the following rules should be obeyed:

- For $m \times n$ matrices A and B, the commutative law of addition holds

$$A + B = B + A$$

- For $m \times n$ matrices A, B and C the associative law of addition holds

$$(A + B) + C = A + (B + C)$$

- For every $m \times n$ matrix A, $n \times p$ matrix B and $p \times q$ matrix C, the associative law of multiplication holds

$$(AB)C = A(BC)$$

- For $m \times n$ matrices A and B and $n \times p$ matrices C and D, the distributive law of multiplication holds

$$\begin{aligned}(A+B)C &= AC+BC\\ A(C+D) &= AC+AD\end{aligned}$$

- Note, however that the commutative law of multiplication does NOT hold for matrices. That is to say that in general for two matrices A and B

$$AB \neq BA$$

Example

$$A = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 4 \\ 5 & 2 \end{pmatrix}$$

Find AB and BA

Solution

$$AB = \begin{pmatrix} 32 & 20 \\ 16 & 10 \end{pmatrix} \neq BA = \begin{pmatrix} 6 & 18 \\ 12 & 36 \end{pmatrix} \tag{6.1}$$

- For every $m \times n$ matrix A, it holds that $I_m A = A I_n = A$. Thus giving the name for the identity matrix I_m and I_n.

Transpose

Let A be a $m \times n$ matrix. Then the $n \times m$ matrix A^T with $(A^T)_{ij} = (A)_{ji}$ is called the transpose of A. A matrix is called symmetric if $A^T = A$ holds.

Example

Determine the transpose of the following matrices

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 & -5 \\ 3 & -1 & 2 \\ -5 & 2 & 7 \end{pmatrix}$$

Solution

$$A^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix} \neq A \text{ is NOT symmetric} \qquad B^T = \begin{pmatrix} 2 & 3 & -5 \\ 3 & -1 & 2 \\ -5 & 2 & 7 \end{pmatrix} = B \text{ is symmetric}$$

The matrix transpose obeys the following rules

- For general $m \times n$ matrices A and B, $(A+B)^T = A^T + B^T$ holds
- For every $m \times n$ matrix A and every $n \times p$ matrix B, $(AB)^T = B^T A^T$ holds.

6.3.3 Matrix notation for linear systems of equations

We are now in a position to write the linear equation system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{6.2}$$

in a much shorter way. To enable us to do this, we define the matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

and the column vectors

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \tag{6.3}$$

The matrix A is called the coefficient matrix and b is called the right hand side of the linear equation system. The equation system(6.2) is equivalent to the matrix equation

$$Ax = b \tag{6.4}$$

To solve this linear system of equations we use the Gauss elimination method discussed earlier.

6.3.4 Matrix inverse

The matrix inverse only makes sense for square matrices. It is defined as follows: The $n \times n$ matrix X is called the inverse of matrix A if $AX = I_n$. If the matrix A has an inverse, the matrix A is called **invertible** or **regular**, if the matrix has no inverse it is called **singular**. For a regular $n \times n$ matrix A, we denote its inverse by A^{-1}.

Let A and B be invertible $n \times n$ matrices, then

- $A^{-1}A = I_n$
- A^{-1} is invertible and $(A^{-1})^{-1} = A$
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

The following statements are equivalent:

- A is invertible
- The linear equation system $Ax = b$ is solvable for every b
- The linear equation system $Ax = 0$ has only the trivial solution $x = 0$

The matrix inverse is very rarely computed as it is an expensive computation. In theory, one could compute the solution to a $n \times n$ linear equation system $Ax = b$ using the matrix inverse, since $A^{-1}Ax = A^{-1}b$ and $A^{-1}A = I_n$ so that $A^{-1}Ax = I_n x = x = A^{-1}b$. However, this is not recommended and Gauss elimination should be used.

To compute the matrix inverse for a regular $n \times n$ matrix A we proceed as follows: We denote the matrix inverse by X and note that

$$AX = \begin{pmatrix} a^{(1)} & \cdots & a^{(n)} \end{pmatrix} \begin{pmatrix} x^{(1)} & \cdots & x^{(n)} \end{pmatrix} = I_n = \begin{pmatrix} b^{(1)} & \cdots & b^{(n)} \end{pmatrix}$$

where $a^{(1)}, \cdots, a^{(n)}$ are column vectors which make up the columns of matrix A and $x^{(1)}, \cdots, x^{(n)}$ are column vectors which make up the inverse of A. To determine $x^{(1)}, \cdots, x^{(n)}$, we can solve linear systems $Ax^{(1)} = b^{(1)}, \cdots, Ax^{(n)} = b^{(n)}$ for $x^{(1)}, \cdots, x^{(n)}$ where $b^{(1)}, \cdots, b^{(n)}$ are columns of the identity matrix I_n. Then, the inverse of A is given by the matrix whose columns are $x^{(1)}, \cdots, x^{(n)}$.

Example

Determine the inverse of the following matrix

$$A = \begin{pmatrix} 0 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

Solution

We follow a similar procedure to that undertaken when solving linear equations with multiple right hand sides.

x_1	x_2	x_3	1_1	1_2	1_3
0	3	-2	1	0	0
4	-2	1	0	1	0
2	-1	1	0	0	1

x_1	x_2	x_3	1_1	1_2	1_3
2	-1	1	0	0	1
4	-2	1	0	1	0
0	3	-2	1	0	0

x_1	x_2	x_3	1_1	1_2	1_3
2	-1	1	0	0	1
0	0	-1	0	1	-2
0	3	-2	1	0	0

x_1	x_2	x_3	1_1	1_2	1_3
2	-1	1	0	0	1
0	3	-2	1	0	0
0	0	-1	0	1	-2

The solutions are $x^{(1)} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ 0 \end{pmatrix}$, $x^{(2)} = \begin{pmatrix} \frac{1}{6} \\ -\frac{2}{3} \\ -1 \end{pmatrix}$ and $x^{(3)} = \begin{pmatrix} \frac{1}{6} \\ \frac{4}{3} \\ 2 \end{pmatrix}$, thus the matrix inverse is given by $A^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & -\frac{2}{3} & \frac{4}{3} \\ 0 & -1 & 2 \end{pmatrix}$. We can check this as follows

$$A^{-1}A = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & -\frac{2}{3} & \frac{4}{3} \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

6.3.5 Rank of a matrix

The rank of a matrix A is the same as the rank of the linear equation system $Ax = 0$. It is denoted by rank A. To determine the rank of a matrix we undertake the Gauss elimination strategy for the linear equation system $Ax = 0$.

6.3.6 Linear independence

Let us consider a sequence of n column vectors $a^{(1)}, a^{(2)}, \cdots, a^{(n)}$ each of length m. We can construct a $m \times n$ matrix whose columns are these vectors

$$A = \begin{pmatrix} a^{(1)} & a^{(2)} & \cdots & a^{(n)} \end{pmatrix}$$

We say that the vectors $a^{(1)}, a^{(2)}, \cdots, a^{(n)}$ are linearly independent if the linear system $Ax = 0$ has only the trivial solution $x = 0$. If the linear system $Ax = 0$ has a non trivial solution $x \neq 0$ we say that the vectors are linearly dependent.

Linear independence means that each of the vectors $a^{(1)}, a^{(2)}, \cdots, a^{(n)}$ cannot be written as a linear combination of the other vectors. Linearly dependent on the other hand means that the vectors $a^{(1)}, a^{(2)}, \cdots, a^{(n)}$ can be written as a linear combination of each other.

In practise, we perform Gauss elimination on the matrix A to decide whether the vectors are linearly independent or not. In particular for a $m \times n$ matrix we have

- If $r = n$ the vectors are linearly independent.
- If $r < n$ the vectors are linearly dependent.
- If $r = m$ the vectors are called generating.
- If $r = n = m$ the vectors are generating and linearly independent and form a basis.

Example
Determine whether the following vectors are linearly dependent or not

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solution
We form the matrix whose columns are the two vectors

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

Next, we perform Gauss elimination on the system $Ax = 0$

x_1	x_2	1_1
1	0	0
1	0	0
1	0	0

x_1	x_2	1_1
1	0	0
0	0	0
0	0	0

We observe that $r = 1$, $m = 3$ and $n = 2$. This means that $r < n$ so that the system is linearly dependent and not generating.

6.4 Determinants

The determinate of a square matrix is an important aspect of linear algebra. It enables one to characterise whether a matrix is regular or singular. With help of determinants one can discuss linear equation systems. There also lies a connection between determinants and volumes. Further topics of linear algebra such as eigenvalues and eigenvectors require the use of determinants.

6.4.1 Definition and properties

A **determinate** is a number which can be computed from each square matrix A. The number is written as $\det A$ or $|A|$. Below, we illustrate some simple examples

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \tag{6.5}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{6.6}$$

To determine the explicit value for the case given in equation (6.6) we use the result from equation (6.5).

The definition of a determinate is a follows

- For a 1×1 matrix $A = (a)$

$$\det A = |A| = a$$

- Set

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

to be a $n \times n$ matrix with $n \geq 2$. For $i = 1, 2, \cdots, n$ set A_{1i} to be the $(n-1) \times (n-1)$ matrix that one obtains when the first row and the ith column has been deleted. Then the number

$$\det A = |A| = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} - \cdots + (-1)^{n+1} a_{1n}\det A_{1n} \qquad (6.7)$$

is called the determinate of A.

Example

Determine the determinants of the following matrices

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 4 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 4 & 1 & 2 \\ 2 & 0 & 2 & 1 \\ 1 & 0 & 2 & 2 \end{pmatrix}$$

Solution

$$\det A = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 3 \cdot 2 - 2 \cdot 1 = 4$$

$$\det B = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 4 & 1 & 2 \end{vmatrix} \begin{aligned} &= 1\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} \\ &= 1 \cdot 4 - 2 \cdot (-4) + 1 \cdot (-10) = 2 \end{aligned}$$

$$\det C = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 2 \\ 2 & 0 & 2 & 1 \\ 1 & 0 & 2 & 2 \end{vmatrix} \begin{aligned} &= 1\begin{vmatrix} 4 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 & 2 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 4 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 2 \end{vmatrix} \\ &= 1\left[4\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix}\right] + \\ &+1\left[2\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix}\right] \\ &= 1[4 \cdot 2 - 1 \cdot 0 + 2 \cdot 0] + 1[2 \cdot 0 - 4 \cdot 3 + 2 \cdot 0] = -4 \end{aligned}$$

Some important properties of determinants are listed below

- If two row of a square matrix are interchanged, the determinate changes sign

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ a_j & b_j & c_j & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ a_i & b_i & c_i & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = - \begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ a_i & b_i & c_i & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ a_j & b_j & c_j & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

- If a one row is multiplied by a constant factor and added to another row, the determinate remains unaltered

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ a_i + \alpha a_j & b_i + \alpha b_j & c_i + \alpha c_j & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ a_j & b_j & c_j & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ a_i & b_i & c_i & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ a_j & b_j & c_j & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

- If a row of a matrix is multiplied by a factor α, the determinate also becomes multiplied by that factor.

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \alpha a_i & \alpha b_i & \alpha c_i & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \alpha \begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ a_i & b_i & c_i & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

- The determinate of a triangular matrix is equal to the product of the diagonal terms.
- For every $n \times n$ matrix A, it holds that $\det A = \det A^T$
- If the $n \times n$ matrix A is invertible then $\det A \neq 0$ and $\det A^{-1} = \frac{1}{\det A}$

6.4.2 Efficient calculation of determinants

We have seen that the multiples of other rows of a matrix can be added to other rows without altering the determinate of a matrix. Also, we noted that when the matrix is in triangular form, the determinate is just simply the product of the diagonal terms. This means that we can use Gauss elimination to achieve efficient calculation of the determinate. After performing Gauss elimination, the determinate is just simply the product of the diagonal entries. Note that if rows are exchanged during Gauss elimination, the determinate changes sign.

Example
Determine the determinant of the following matrix using Gauss elimination

$$A = \begin{pmatrix} 0 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

Solution

$$\begin{aligned} \det A &= \begin{vmatrix} 0 & 3 & -2 \\ 4 & -2 & 1 \\ 2 & -1 & 1 \end{vmatrix} = - \begin{vmatrix} 2 & -1 & 1 \\ 4 & -2 & 1 \\ 0 & 3 & -2 \end{vmatrix} \\ &= - \begin{vmatrix} 2 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & -1 \end{vmatrix} \end{aligned}$$

The determinate is then the product of the diagonal terms $\det A = 2 \cdot 3 \cdot (-1) = -6$

6.4.3 Determinants and linear equation systems

The first thing to note is that when A is a $n \times n$ matrix, then performing the Gauss elimination procedure leads one to the conclusion that $\det A \neq 0$ exactly when a matrix has full rank ($r = n$).

Furthermore the following statements are equivalent:

- The matrix A is invertible
- $\det A \neq 0$
- After performing Gauss elimination $r = n$
- The linear equation system is solvable for every b.
- The solution of the linear equation system $Ax = b$ is unique.
- The linear equation system $Ax = 0$ has only the trivial solution $x = 0$.

If we set A to be a $n \times n$ matrix, then the following holds

- The homogeneous linear equation system $Ax = 0$ has only the trivial solution when $\det A \neq 0$.
- The linear equation system $Ax = b$ is solvable for any right hand sides when $\det A \neq 0$.
- The solution of the linear equation system $Ax = b$ is unique when $\det A \neq 0$.

Now if we consider the solution set of a linear equation system with n equations and n unknowns we have the following

- If $\det A \neq 0$ the homogeneous linear equation system $Ax = 0$ has only the trivial solution.
- If $\det A = 0$ the homogeneous linear equation system $Ax = 0$ has infinitely many solutions.
- If $\det A \neq 0$ the linear equation system $Ax = b$ has for a general right hand side vector exactly one solution.
- If $\det A = 0$ the linear equation system $Ax = b$ has no solution or infinitely many solutions, depending on the right hand side vector.

6.5 Eigenvalue Problems

The eigenvalue problem is one of the most important exercises in linear algebra. In what follows we shall describe how eigenvalues and eigenvectors may be computed by hand for small matrices.

6.5.1 Eigenvalues

Let us consider a $n \times n$ matrix A. We now ask the question, does a a (column) vector exist that allows us to write the matrix vector product Ax in a particularly simple way? In other words, is there a vector such that we can write Ax as a number multiplied by x?

Eigenvalues and eigenvectors are defined as follows

- The number λ is called an **eigenvalue** of matrix A, if there exists a vector x such that $Ax = \lambda x$ holds.
- If λ is an eigenvalue of the matrix A, then the vector x, for which $Ax = \lambda x$ holds, is called the **eigenvector** of matrix A corresponding to eigenvalue λ.

For the moment, we just want to characterise the eigenvalue of matrix A. As we described above, the value λ is an eigenvalue of A when there is a vector $x \neq 0$ such that $Ax - \lambda x = 0$ holds. The equation can also be written as $Ax - \lambda I_n x = 0$ or $(A - \lambda I_n)x = 0$. The number λ is therefore the eigenvalue of the matrix A when the homogeneous equation system $(A - \lambda I_n)x = 0$ has a non–trivial solution. It follows from the previous section, that this is exactly the case when $\det(A - \lambda I_n) = 0$.

Example

Determine the eigenvalues of the following matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

Solution

$$A - \lambda I_n = \begin{pmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{pmatrix}$$

Next, we compute the determinate of this matrix

$$\begin{aligned} \det(A - \lambda I) &= (-2-\lambda)\left[(-2-\lambda)^2 - 1\right] - 1\left[(-2-\lambda) - 0\right] \\ &= (-2-\lambda)\left[(-2-\lambda)^2 - 2\right] = -(2+\lambda)(\lambda^2 + 4\lambda + 2) \end{aligned}$$

The cubic equation $\det(A - \lambda I) = 0$ has the following roots, $\lambda_1 = -2$, $\lambda_2 = -2 + \sqrt{2}$ and $\lambda_3 = -2 - \sqrt{2}$ which are also in turn the eigenvalues of matrix A.

In general for a $n \times n$ matrix A we observe that $\det(A - \lambda I_n)$ is a polynomial of nth degree in λ. We call the polynomial $\det(A - \lambda I_n)$ the **characteristic polynomial** of matrix A and denote it by $P_A(\lambda)$. If the polynomial $P_A(\lambda)$ has a root λ^* which is repeated k times, we call k the **algebraic multiplicity** of eigenvalue λ^*.

We have the following properties

- Every $n \times n$ matrix has at least one eigenvalue.
- Every $n \times n$ matrix has at most n eigenvalues.

- The algebraic multiplicity of every eigenvalue is greater or equal to 1 and less than or equal to n.
- Every $n \times n$ matrix has exactly n eigenvalues when the algebraic multiplicity of each eigenvalue is taken in to account.
- For every real matrix the coefficients of the characteristic polynomial are real. In this case, the eigenvalues are either real or appear in complex conjugate pairs.
- The following holds for the characteristic polynomial $P_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$

$$\begin{aligned} c_n &= (-1)^n \\ c_{n-1} &= (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}) \\ c_0 &= \det A \end{aligned}$$

Example
Determine the eigenvalues of the following matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Solution
This time we use Gauss elimination to compute det $(A - \lambda I_n)$

$$\begin{aligned} \det(A - \lambda I_n) &= \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 2-\lambda \\ 1 & 2-\lambda & 1 \\ 2-\lambda & 1 & 1 \end{vmatrix} \\ &= - \begin{vmatrix} 1 & 1 & 2-\lambda \\ 0 & 1-\lambda & -1+\lambda \\ 0 & -1+\lambda & 1-(2-\lambda)^2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 2-\lambda \\ 0 & 1-\lambda & -1+\lambda \\ 0 & 0 & -4+5\lambda-\lambda^2 \end{vmatrix} \\ &= (1-\lambda)(4-5\lambda+\lambda^2) \end{aligned}$$

Therefore we have $P_A(\lambda) = -(\lambda - 1)(4 - 5\lambda + \lambda^2) = -(\lambda - 1)^2(\lambda - 4)$. Therefore there are two eigenvalues 1 and 4.
The eigenvalue $\lambda = 1$ has algebraic multiplicity 2.
The eigenvalue $\lambda = 4$ has algebraic multiplicity 1.

6.5.2 Eigenvectors

Let us set A to be an $n \times n$ matrix and λ to be an eigenvalue of this matrix. We have seen, that when the determinate of the matrix $(A - \lambda I_n)$ is equal to zero, there exists an eigenvector $x \neq 0$ to matrix A corresponding to the eigenvalue λ, if

$$(A - \lambda I_n)x = 0 \tag{6.8}$$

The set of eigenvectors corresponding to eigenvalue λ is equal to set the set of non–trivial solutions to the equation system (6.8). We call this set of nontrivial solutions the **eigenspace of** A **corresponding to eigenvalue** λ and is given the symbol E_λ. The dimension of E_λ is called the **geometric multiplicity** of the eigenvalue λ. The geometric multiplicity is always greater or equal to 1.

The **span** is the set of all linear combinations of a set of vectors which make up the eigenspace.

Example

Given the following matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

For which we have already found that its eigenvalues are $\lambda_1 = -2$, $\lambda_2 = -2 + \sqrt{2}$ and $\lambda_3 = -2 - \sqrt{2}$, now compute the corresponding eigenspaces

Solution

Eigenspace for $\lambda = -2$. The coefficient matrix of the linear equation (6.8) is

$$A + 2I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

With Gauss elimination we can find the eigenspace E_{-2}

x_1	x_2	x_3	1
0	1	0	0
1	0	1	0
0	1	0	0

x_1	x_2	x_3	1
1	0	1	0
0	1	0	0
0	1	0	0

x_1	x_2	x_3	1
1	0	1	0
0	1	0	0
0	0	0	0

The solution set is $\{x_3 = \alpha, x_2 = 0, x_1 = -\alpha | \alpha \in \mathbb{R}\}$ or

$$E_{-2} = \left\{ \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Eigenspace to $\lambda = -2 + \sqrt{2}$. Proceeding with Gauss elimination we have

x_1	x_2	x_3	1
$-\sqrt{2}$	1	0	0
1	$-\sqrt{2}$	1	0
0	1	$-\sqrt{2}$	0

x_1	x_2	x_3	1
1	$-\sqrt{2}$	1	0
$-\sqrt{2}$	1	0	0
0	1	$-\sqrt{2}$	0

x_1	x_2	x_3	1
1	$-\sqrt{2}$	1	0
0	-1	$\sqrt{2}$	0
0	1	$-\sqrt{2}$	0

x_1	x_2	x_3	1
1	$-\sqrt{2}$	1	0
0	-1	$\sqrt{2}$	0
0	0	0	0

The solution has the form $x_3 = \alpha$, $x_2 = \sqrt{2}\alpha$, $x_1 = \alpha$, $\alpha \in \mathbb{R}$. Thus

$$E_{-2+\sqrt{2}} = \left\{ \alpha \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \right\}$$

Eigenspace to $\lambda = -2 - \sqrt{2}$. In a similar fashion to the above, we get

$$E_{-2-\sqrt{2}} = \left\{ \alpha \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \right\}$$

Example
Given the following matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

For which we have already found that its eigenvalues are $\lambda_1 = \lambda_2 = 1$, and $\lambda_3 = 4$, now compute the corresponding eigenspaces

Solution

Eigenspace for $\lambda = 1$. With help of Gauss elimination we have

x_1	x_2	x_3	1
1	1	1	0
1	1	1	0
1	1	1	0

x_1	x_2	x_3	1
1	1	1	0
0	0	0	0
0	0	0	0

The solution set is $\{x_3 = \alpha, x_2 = \beta, x_1 = -\alpha - \beta | \alpha, \beta \in \mathbb{R}\}$, thus

$$\begin{aligned} E_1 &= \left\{ \begin{pmatrix} -\alpha - \beta \\ \beta \\ \alpha \end{pmatrix} | \alpha, \beta \in \mathbb{R} \right\} = \left\{ \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} | \alpha, \beta \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

This solution set has two free parameters. The dimension of E_1 is therefore 2, we say that the geometric multiplicity of eigenvalue $\lambda = 1$ is 2.

Eigenspace for $\lambda = 4$. With help of Gauss elimination we have

x_1	x_2	x_3	1
-2	1	1	0
1	-2	1	0
1	1	-2	0

x_1	x_2	x_3	1
1	-2	1	0
0	-3	3	0
0	3	-3	0

x_1	x_2	x_3	1
1	-2	1	0
0	-3	3	0
0	0	0	0

The solution has the form $x_3 = \alpha$, $x_2 = \alpha$, $x_1 = \alpha$, $\alpha \in \mathbb{R}$

$$E_4 = \left\{ \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} | \alpha \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

The geometric multiplicity of $\lambda = 4$ is 1.

6.6 Summary and Further Reading

The key topics from this chapter and references to further reading in James (fourth edition), James (customised edition) and Croft and Davison (third edition) are

- Gauss elimination. Croft and Davison [pg 573-585]. James (customised and fourth edition) [pg 356-371].
- Matrix definitions, multiplication etc. Croft and Davison [pg 483-501]. James (customised and fourth edition) [pg 297-328].

- Using the matrix inverse to solve linear equations. NOTE THIS IS NOT A RECOMMENDED APPROACH!. Croft and Davison [pg 561-572,521-529]. James (customised and fourth edition) [pg 341-350].
- Determinates. Croft and Davison [pg 502-520]. James (customised and fourth edition) [pg 328-340].
- Eigenvalues and Eigenvectors. Croft and Davison [pg 586-601]. James (customised and fourth edition) [pg 387-407].