# Swansea University
# Mathematics scholarship exam 2013

2 hours 30 minutes
Calculators allowed, but no formula books.

*Please attempt all the questions in section A, and then at most four from section B. Explanations of your solutions will form part of the assessment.*

## Section A

1. Write the repeating decimal expansion $0{\cdot}3(14)^{\bullet}$ as a fraction with integer numerator and denominator (e.g. something like 7/13). Write the fraction 7/11 as a repeating decimal expansion.

2. Determine whether 437 is prime or not. Show your reasoning.

3. For what real values of $x$ is the following inequality true?

$$\frac{x^3-1}{(x+1)^2} > 0$$

4. The number of bacteria in a culture increases by 5% every minute. If there were initially two million bacteria in the culture, how many would there be after 270 seconds?

5. Evaluate the integral

$$\int_0^2 \sqrt{4-x^2}\,\mathrm{d}x\ .$$

6. There is one real number $x$ which satisfies $e^x = 2 - x$ and $0 \le x \le 1$. Calculate $x$ to within an error of $\frac{1}{16}$.

7. Find the differential with respect to $x$ of

$$\frac{\sin(x)}{1+x^2}\ .$$

8. Evaluate the integral

$$\int_0^1 \frac{x\ \mathrm{d}x}{x^2+4}\ .$$

9. A triangle has sides 4 and 5, and the angle between those sides is $60°$. What is the length of the third side?

10. Given that $x = 1$ is a root of the cubic $x^3 + 2x^2 - 2x - 1$, find the other two roots.

## Section B

**1.** The function $f(x)$ of a real variable $x$ is given by the formula $f(x) = 1 + x^{-1}$.

a) Find the single real $\gamma > 0$ so that $f(\gamma) = \gamma$.

b) Show that if $x > \gamma$ then $0 < f(x) < \gamma$.

c) Show that if $0 < x < \gamma$ then $f(x) > \gamma$.

d) Show that if $0 < x < \gamma$ then $x > f(f(x)) > \gamma$.

**2.** The world record for the shot put is held by the American athlete Randy Barnes at 23.12 metres. His record has stood since since 1990.

A mens shot put weighs 7.2kg. Taking $g = 9.8ms^{-2}$, find how fast an athlete would need to launch the shot put to break the world record?

*Hint:* For simplicity you may ignore the height of the athlete and assume that the shot put is launched from on the floor.

**3.**

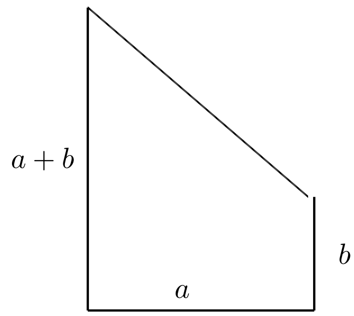
a) State the principle of induction (any version will do).

b) The sequence of integers $a_n$ for $n \geq 1$ is given by $a_1 = 1$ and the recursive equation $a_{n+1} = a_n + 2\,n + 2$ for $n \geq 1$. Write down the values of $a_2$ and $a_3$.

c) Find values of constants $A$ and $B$ so that $a_n = n^2 + A\,n + B$ fits the first few values of $a_n$.

d) Show by induction that the formula for $a_n$, with your determined values of $A$ and $B$, is a solution of the recursive equation in (a) for all integer $n \geq 1$.

4. A metal plate has four sides, one (length $a$ metres) horizontal, two (length $b$ and $a+b$ metres) vertical, and the fourth at 45° to the others.




a) What is the area of the plate? What is the length of the perimeter of the plate?

b) The length of the perimeter of the plate is fixed at 2 metres. Find a formula for $b$ in terms of $a$ which ensures that the length of the perimeter has this value.

c) Given that the length of the perimeter of the plate is fixed at 2 metres, find the value of $a$ which maximises the area of the plate, and the maximum area of the plate.

5. A particle of mass $m$ rests on a horizontal plank of wood. The coefficient of friction between the particle and the plank is $\mu$. The plank is slowly inclined until the particle is on the point of slipping at which point the angle between the plank and the horizontal is given by $\theta$.

The plank is then replaced by a second much rougher plank. The coefficient of friction between the particle and the new plank is $3\mu$. Again the plank is inclined until the particle is on the point of slipping at which point the angle between the plank and the horizontal is $2\theta$.

Find the values of $\theta$ and $\mu$.

6. Inspector Code of the Swansea police is investigating a crime on the island of Knights and Knaves. Remember that Knights always tell the truth, that Knaves always lie, and that all the inhabitants of the island are either Knights or Knaves.

(a) The inspector meets an inhabitant who says 'If I am a Knight, then I am guilty'. Is the inhabitant guilty, or is there insufficient information to tell?

(b) The inspector meets another inhabitant who says 'If I am a Knave, then I am guilty'. Is the inhabitant guilty, or is there insufficient information to tell?

(c) The inspector meets two more inhabitants, P and Q, who make statements:

P says: If I am guilty, then both of us are guilty.
Q says: Exactly one of us is innocent.
P says: Q is innocent.

State whether Q is innocent or guilty, and also whether Q is a Knight or a Knave.

(d) The inspector meets another inhabitant who says 'If I am guilty, then I am a Knight'. Inspector Code replies 'you expect me to say that I cannot tell if you are innocent or guilty. However I was given a note by the authorities on the island earlier, telling me whether you were a Knight or a Knave. As a result of that and what you just said, I know definitively whether you are innocent or guilty'. Which is the case, is the inhabitant innocent or guilty?

7. The Gaulish chieftain Combinatorix is going to Rome to see Julius Cæsar, and is taking four druids and six warriors with him.

(a) On the journey to Rome the party walks in single file, with Combinatorix always at the front. Given that the warriors and druids can all be told apart, how many different ways can the party be arranged in a line?

(b) They come to a village with two inns and stay for the night. Combinatorix takes the finest room, leaving the rest of the party to be split between the first inn, which has six places, and the second inn, which has 4 places. How many different ways can this splitting between the two inns be done?

(c) The next night they come to another village with two inns and stay for the night. Combinatorix again takes the finest room, leaving the rest of the party to be split between the first inn, which has six places, and the second inn, which also has 6 places. In how many different ways can this splitting between the two inns be done?

(d) The party stays at the village in (c) for a second night. The two innkeepers ask Combinatorix if he can arrange for exactly three warriors to stay in each inn, as they are afraid that too many warriors in one inn could cause trouble. How many different ways can this splitting between the two inns be done, given the restriction on placing the warriors?

(e) On leaving the village, the innkeepers present the four druids in the party with new robes. These robes are identical, and it is now impossible to tell the druids apart. The party again walks in single file, with Combinatorix leading the way, as they leave the village. In how many different distinguishable ways can the party be arranged in a line, given that all the druids now look the same?

8. Two wooden blocks of thickness $2a$ and $4a$ respectively are held fixed next to each other. A gun fires a pellet of mass $m$ horizontally at the blocks so that the pellet hits the first block perpendicular to its surface. Suppose that as the bullet passes through the first block its motion is resisted by a force $F_1$ and that as it passes through the second block its motion is resisted by a force $F_2$.

a) The bullet hits the first block with a speed $u$ and enters the second block with a speed $v$. It is bought to rest after travelling a distance $a$ into the second block. Find $F_1$ and $F_2$ in terms of $u, v, m, a$.

b) A second bullet of mass $m$ is now fired in the opposite direction so it hits the second block first with a speed $u$. Show that this bullet will pass through both blocks if $\sqrt{5}v < u$.

9. All dice are six sided, with sides labelled 1,2,3,4,5,6 as usual, and are assumed to be fair. All throws of the dice are assumed to be independent.

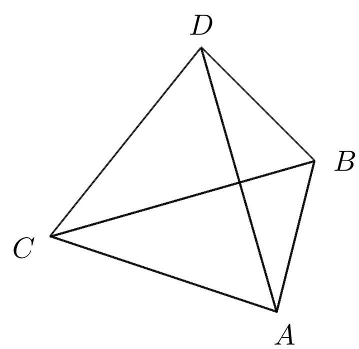
(a) Two six sided dice are thrown, and the numbers on the sides are added. What is the probability that the sum is 7? What is the probability that the sum is 6?

(b) Two six sided dice are thrown by another person, who tells you that the sum of the dice is 6 (you cannot see the dice). Given that information, what is the probability that both dice were 3?

(c) Two six sided dice are thrown. If the numbers on both dice are the same, and only in this case, a third dice is thrown. What is the probability that the sum of all the dice is 7? What is the probability that the sum is 6?

(d) The game in (c) is played by another person, and you cannot see the dice. That person says that the total on all the dice thrown was 7. Given that information, what is the probability that three dice were thrown?

10. A regular tetrahedron is a three dimensional figure with four faces which are all equilateral triangles. In the tetrahedron below the points $ABCD$ are all connected with metal rods of length 1 metre. It stands on the horizontal base $ABC$. The faces are covered with sheets of paper.




(a) Let M be the mid-point of the line AB. What is the length of the line CM?

(b) What is the total surface area of the tetrahedron (including the base).

(c) By considering the triangle CMD, find the angle that the line CD makes to the horizontal.
(d) What is the height of the tetrahedron, i.e. how far is D above the base ABC? You might find the triangle CMD in (c) useful.