Swansea University
Mathematics scholarship exam 2012

2 hours 30 minutes
Calculators allowed, but no formula books.

Please attempt all the questions in section A, and then at most four from section B. Explanations of your solutions will form part of the assessment.

Section A

1. Write the repeating decimal expansion $0{\cdot}3(15)^{\bullet}$ as a fraction with integer numerator and denominator (e.g. something like 7/13). Write the fraction 5/7 as a repeating decimal expansion.

2. Determine whether 367 is prime or not. Show your reasoning.

3. For what real values of x is the following inequality true?

$$\frac{(x-1)^2}{x^2-16} \geq 0$$

4. The number of bacteria in a culture increases by 5% every minute. If there were initially one million bacteria in the culture, how many would there be after 150 seconds?

5. Evaluate the integral

$$\int_0^1 \frac{1}{x^2+3\,x+2}\,\mathrm{d}x\ .$$

6. There is one real number x which satisfies $\cos(x) = x$ and $0 \leq x \leq 1$. Calculate x to within an error of $\frac{1}{16}$. (Remember to use radians when evaluating cosine).

7. Find the differential with respect to x of $\cos(1/x^2)$.

8. Evaluate the integral

$$\int_0^1 \frac{dx}{x^2+4}\ .$$

9. What is the area of a triangle with sides 2, 4 and 5?

10. Given that $x = 2$ is a root of the cubic $x^3 + 3\,x^2 - 5\,x - 10$, find the other two roots.

Section B

1. A Pythagorean triple is a list of three strictly positive integers a, b, c satisfying $a^2 + b^2 = c^2$. For any two integers n, m satisfying $n > m > 0$, there is a Pythagorean triple given by

$$a = n^2 - m^2, \qquad b = 2nm, \qquad \text{and} \qquad c = n^2 + m^2.$$

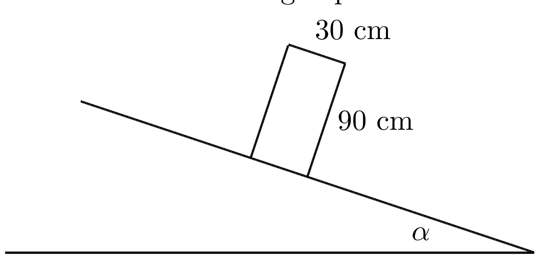
In what follows, you may assume that if a prime p divides x^2 for some integer x, then it must divide x.

a) Use the formula to find four Pythagorean triples, where a, b, c are coprime (i.e. have no common factors except 1).

b) Suppose that n and m are coprime (i.e. have no common factors except 1), and that one of n, m is odd and the other is even. Show that $a = n^2 - m^2$ and $b = 2\,n\,m$ have no common factors except 1.

c) Suppose we now want to find integers d, e, f, g that satisfy $d^3 + e^3 + 2\,f^3 = g^3$. (Note the coefficient 2 in the equation!) Can you adapt the formula for Pythagorean triples to find a formula that will give infinitely many different such d, e, f, g? Hint: Look at $d = k\,n^3 - m^3$ and $g = k\,n^3 + m^3$, where $k \neq 0$ is a small integer chosen to make the formula work.

2. A rectangular block of mass 4 Kg is placed on an inclined plane as shown below:



The angle α is slowly increased, starting at $0°$.
Does the block slide, or does it topple over, if the coefficient of friction $\mu = 0{\cdot}4$?

3.

a) State the principle of induction (any version will do).

b) The sequence of integers a_n for $n \geq 1$ is given by $a_1 = 1$ and the recursive equation $a_{n+1} = 3\,a_n + 4$ for $n \geq 1$. Write down the values of a_2 and a_3.

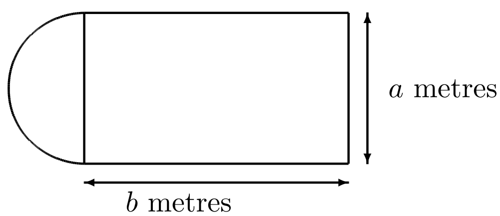
c) Show by induction that the value of a_n in part (b) is given by the formula $a_n = 3^n - 2$.

A garden path is rectangular; its width is 2 feet (2') and its length is a whole number, n, of feet. It is to be paved using paving stones that are rectangular and 2' by 1' in size. Let b_n denote the number of different ways of arranging the paving stones to make the path.

d) Calculate b_1, b_2 and b_3.

e) Find a formula that expresses b_{n+2} in terms of (some of) the lower values $b_1, b_2, \ldots, b_{n+1}$.

4. A metal plate is formed by welding together a semicircular plate and a rectangular plate, as shown in the diagram. The sides of the rectangle are marked by length in metres as a and b.



a) What is the area of the plate? What is the length of the perimeter of the plate (do not count the length of the weld joining the plates). Your answers should be expressed in terms of a and b.

b) The length of the perimeter of the plate is fixed at $4 + \pi$ metres. Find a formula for b in terms of a which ensures that the length of the perimeter has this value.

c) Given that the length of the perimeter of the plate is fixed at $4 + \pi$ metres, find the value of a which maximises the area of the plate.

5. An object of mass m moving with speed $3\,u$ hits an object of mass $3\,m$ moving with speed u in the opposite direction. Show that the ratio of the speeds remains unchanged after the collision (assuming that the coefficient of restitution is non-zero).

6. Inspector Code of the Swansea police is investigating a robbery. Only three people, X, Y and Z, could have committed the crime. The following facts are known:

i) If X is innocent, then Y is also innocent.

ii) If Z was not involved, then exactly one person committed the crime.

iii) Either Y acted alone, or exactly two people were involved in the crime.

a) Given the information (i,ii,iii), which combination of X, Y and Z committed the robbery?

On the morning of the trial, Inspector Code receives an urgent message about new evidence in the case. There has been a mistake at the forensic laboratory, and one of the facts (i,ii,iii) is no longer known to be correct. Inspector Code tells his assistant 'because of the mistake, we can now only convict one of the suspects, the other two will have to be released because of lack of evidence, although they may be guilty'.

b) Which of the previous 'facts' (i,ii,iii) is no longer known to be correct? Which suspect can still be convicted?

7.

(a) A jeweller arranges six different coloured beads (blue, red, green, white, yellow, black) on a straight piece of wire, and displays it in the shop window. In how many different ways can they be arranged?

(b) Now they are arranged on a circular piece of wire, which is again displayed in the shop window. As the circle has no obvious beginning, there is no obvious first place to start the counting from. For example the arrangement (blue, red, green, white, yellow, black) would be counted the same as the arrangement (black, blue, red, green, white, yellow). In how many different ways can they be arranged on the circle?

(c) The next day, the blue and red beads have been lost. To make up the number to six, two extra black beads are added, so that we now have (green, white, yellow, $3\times$black). In how many different ways can they be arranged on the straight piece of wire? Note that the black beads are identical, so that they cannot be told apart.

(d) In how many different ways can the beads (green, white, yellow, $3\times$black) be arranged on the circular wire?

8. A toy parachutist is thrown upwards from ground level with a speed $5\,g$. It rises to its highest point, and then begins to fall back to the ground. After it has fallen for one second, the parachute opens. When this happens the toy does not decelerate, but it stops accelerating, and continues downward at a constant speed. How long does the toy take to fall from its highest point? [The acceleration due to gravity is $g\ m/s^2$.]

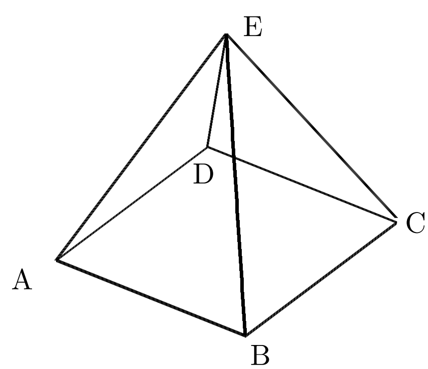
9. A vase initially contains 4 black balls and 6 white balls. Apart from colour, the balls are identical. Balls are taken from the vase without looking, so the choice is random.

(a) A ball is taken from the vase, its colour is noted, and it is returned. The vase is shaken to mix the balls. Then another ball is taken from the vase. What is the probability that both balls were white? What is the probability that the balls were different colours?

(b) A ball is taken from the vase, its colour is noted, and it is **not** returned, leaving 9 balls in the vase. The vase is shaken to mix the balls. Then another ball is taken from the vase. What is the probability that both balls were white? What is the probability that the balls were different colours?

(c) A ball is taken from the vase, its colour is noted, and two balls of that colour are put in the vase (so the vase now has a total of 11 balls). What is the probability that both balls were white? What is the probability that the balls were different colours?

10. A square based pyramid (with base the horizontal square ABCD in the diagram) has apex (highest point) E, which is directly above the midpoint of the square. The length of AB is 50 metres, and the point E is a height 35 metres above the midpoint of the base square.



(a) Let M be the mid-point of the line AB. What is the length of the line EM?

(b) What is the total surface area of the pyramid (including the base).

(c) What is the length of the line AE?

(d) What is the volume of the pyramid? [If you don't know how to work out the volume of a pyramid, do not spend ages trying to do it!]