Swansea University
Mathematics scholarship exam 2011

2 hours 30 minutes
Calculators allowed, but no formula books.

Please attempt all the questions in section A, and then at most four from section B. Explanations of your solutions will form part of the assessment.

Section A

1. Write the repeating decimal expansion $0{\cdot}4(13)^{\bullet}$ as a fraction with integer numerator and denominator (e.g. something like 7/13). Write the fraction 8/11 as a repeating decimal expansion.

2. Determine whether 551 is prime or not. Show your reasoning.

3. For what real values of x is the following inequality true?

$$\frac{(x-1)\,(x^2-4)}{x-3} \geq 0$$

4. Liquid stored in a barrel escapes continuously at the rate of 7% a year. [This means that there is only 93% of the liquid left after one year.] If there was originally 300 litres in the barrel, how much will be left in $4\frac{1}{2}$ years?

5. Evaluate the integral

$$\int_0^1 x\, e^{x^2}\, \mathrm{d}x \ .$$

6. Find the solution $x \geq 0$ (there is only one in this region) to the following equation correct to within an error of $\frac{1}{16}$:

$$x^3 \ = \ 2\,x + 2 \ .$$

7. Find the differential with respect to x of $\sin(1/x)$.

8. Evaluate the integral

$$\int_0^1 \frac{dx}{x^2 + 2\,x + 2}\ .$$

9. What is the area of a triangle with sides 2, 3 and 4?

10. Given that $x = -1$ is a root of the cubic $x^3 + 4\,x^2 + 4\,x + 1$, find the other two roots.

Section B

1. Suppose that a fraction x can be written in the following form, as a repeating decimal with the repeat unit of length four,

$$x = 0{\cdot}(abcd)^{\bullet} = 0{\cdot}abcdabcdabcdabcd\ldots ,$$

where a, b, c, d are decimal digits (i.e. one of $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$). For example, if $abcd = 6102$ we get

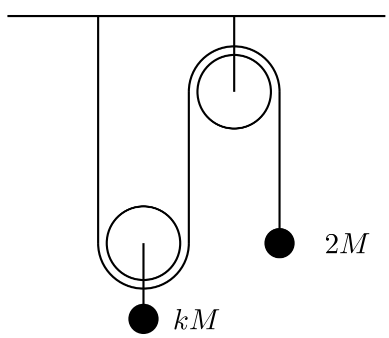
$$\frac{678}{1111} = 0{\cdot}61026102610261026102\ldots .$$

a) Find a formula for $x = n/m$ as a ratio of integers (as in the example) in the general case $abcd$.

b) If x can be written as $x = n/m$ in lowest terms, show that m must divide 9999.

c) Show that the only cases where $x = 1/p$ for p a prime number which can occur as a repeating decimal of our given form are for $p = 3$ or $p = 11$ or $p = 101$.

2. One end of a light inextensible string is attached to the ceiling. The string passes under a smooth light pulley which is carrying a mass kM. The pulley is free to move and is suspended only by the string. The string then passes over a fixed light smooth pulley and at the other end a particle of mass $2M$ is attached. The system is released from rest. Find the tension in the string and the acceleration of the movable pulley in terms of k, M and g where g is the acceleration due to gravity. Show that the particle of mass $2M$ will ascend if $k > 4$.



3.

a) State the principle of induction.

b) Prove the following, for all integer $n \geq 2$,

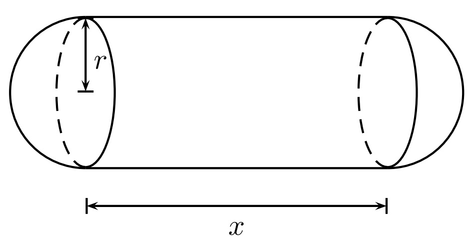
$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{(n-1)\times n} = \frac{n-1}{n} .$$

c) From (b) deduce that for all integer $n \geq 2$,

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} < 1 .$$

d) The polynomial $p(n) = n^2 - n + 41$ has the property that $p(n)$ is prime for all integer values of n in the range $0 \leq n \leq 20$, and for more values of n than that. Can you show that $p(n)$ is not prime for some $n > 20$? Hint: Do not try values starting at $n = 21$ and see if they are prime or not - this might take a long time. Look for an obvious value of n to try.

4. A metal gas container is made by taking a circular cylinder of radius r meters and length x meters, and attaching a hemisphere to both ends, as in the following diagram.



a) What is the volume of the container in terms of r and x?

b) Because of the difficulty of manufacturing curved metal plates, the cost of the hemispherical plates is £4 per square meter, wheras the cost of the cylindrical plates is only £1 per square meter. What is the total cost of the metal plates for the container in terms of r and x?

c) Given that the volume of the container is $5\pi/3$, find a formula for x in terms of r.

d) Given that the volume of the container is $5\pi/3$ (as in (c)), find the values of r and x which minimise the cost of the metal plates.

[All lengths are given in meters m, areas in m^2 and volumes in m^3. You can use the fact that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, and its area is $4\pi r^2$.]

5. A curve is given as the locus of points (x, y) on the xy-plane which are the same distance from the point $F = (0, 8)$ as from the x axis. [Remember that the distance from a point to a line is measured perpendicular to the line.]

a) Give the equation of the curve in the form where y is given as a formula in x.

b) Give the equation of the line normal to the curve through a point $P = (x, y)$ on the curve.

c) Show that the angle between the line from F to P and the normal to the curve at P is the same as the angle between the normal to the curve at P and a vertical line through P. [By vertical line, I mean parallel to the y axis.]

6. A logician visits the island of Knights and Knaves. Remember that Knights always tell the truth, that Knaves always lie, and that every inhabitant is either a Knight or a Knave.

(a) On one occasion the logician meets two inhabitants A and B. He has no previous knowledge of whether either of the inhabitants is a Knight or a Knave. To find out something about this, he asks B the question 'is at least one of you a Knave?'. The reply from B is either 'Yes' or 'No', but because of the noise of a passing train you cannot hear which. However the logician did hear the reply, and remarks 'now I know exactly what A and B are'. Can you work out whether the answer was 'Yes' or 'No'? What are A and B?

(b) On another occasion the logician meets two inhabitants C and D while he is walking to the post office. The meeting takes place at a fork in the road, with two possible ways forward, left or right, and only one of these leads to the post office. The logician asks which way to go to get to the post office.

D says 'C is a Knight'.

C says 'if the way is left, then D is a Knave'.

Which way is the post office? Can you identify C and D?

7. [Note that each answer does not depend on the previous part in this question, so if you can't do one part, look at the next part.] On the occasion of a royal tournament at the court of King Arthur, there was an entry of 10 knights. The first thing they all had to do was to ride past the king in single file.

(a) How many possible orders were there for the knights to ride past in?

Next the heralds organised the knights into pairs for jousting. Each pair was labelled with the time at which its joust was to take place.

(b) In how many possible ways could they have done this pairing?

As each contest was completed, the heralds announced the result to the crowd.

(c) Given that we now know who was in each pair, how many possible results could there be from the whole days competition of 5 jousts (one for each pair). Note that a draw was not an acceptable outcome.

When the jousting was done for the day, the knights (fortunately still 10 of them) sat round the round table to dine. All the seats at the round table were identical, except one particularly fine one, which was reserved for the knight who had done best. That day, this place went to Sir Lancelot.

(d) How many possible ways could the rest of the knights be arranged round the table?

On the next day, the knights were split into two teams of equal numbers, one team being given blue armour, and the other red.

(e) How many possible ways could this splitting have been carried out?

8. Two balls are projected from the same point with the same initial speed. The angles of projection are 2α and α to the horizontal respectively. Suppose that there is a delay of time T between the first ball being projected and the second ball being projected. Given that the balls collide find the speed of projection in terms of T and α.

9. Two fair 6-sided dice (with faces labelled 1,2,3,4,5,6) are thrown one after the other. What is the probability of the following two events?

(a) The sum of both dice is 7.

(b) The first dice is even, and the sum of both dice is 7.

Now we start another experiment. Throw a 6-sided dice. If the result is even, throw a second dice. If the result is odd, no other dice is thrown.

(c) What is the probability that the total is 5 (i.e. the result is 5 if just one was thrown, and the sum is 5 if both were thrown).

(d) What is the probability that we only have 5s or 6s in the result. I.e. one dice is a 5 or 6 if just one was thrown, and both are 5 or 6 if both were thrown (we do not require that both need be the same - one could be a 5 and the other a 6).

10. A particle of mass m rests on a rough plane which is inclined at an angle θ to the horizontal. The coefficient of friction between the plane and the particle is μ. A horizontal force of magnitude P is applied to the particle and is just sufficient to prevent the particle from sliding down the slope. If however a force of magnitude $2P$ is applied parallel to the slope and acting up the slope, then the particle is just on the point of slipping up the slope. If instead a particle of mass $2m$ sits on the slope then a force $2P$ up the slope is just sufficient to prevent the particle from sliding down. Find the values of θ, μ and P in terms of m and g where g is the acceleration due to gravity.