

## Regular polygons and Fermat primes

A Fermat prime is a prime number  $p$  of the form  $p = 2^n + 1$ . If  $m$  is odd then

$$a^m + 1 = (a + 1)(a^{m-1} - a^{m-2} + a^{m-3} - a^{m-4} + \dots + a^2 - a + 1).$$

For instance

$$a^5 + 1 = (a + 1)(a^4 - a^3 + a^2 - a + 1).$$

Putting  $a = 2^r$  we see that if  $m$  is odd then  $2^{rm} + 1$  is divisible by  $2^r + 1$ . This means that if  $n$  has an odd factor  $k$ , then  $2^n + 1$  is divisible by  $2^k + 1$ . So if  $2^n + 1$  is to be prime, then  $n$  must itself be a power of 2.

There are only five known Fermat primes: 3, 5, 17, 257 and 65537. If there are any more then the smallest has more than 80 digits, and so is greater than the number of atoms in the observable universe.

Gauss proved at the age of 18 that it is possible, using a ruler and compass only, to draw a regular polygon with  $N$  sides if and only if the number  $N$  satisfies both these two conditions:

- (A) each of the odd prime factors of  $N$  is a Fermat prime
- (B)  $N$  is not divisible by the square of any Fermat prime.

For instance  $N = 60$  is possible, since  $60 = 2^2 \times 3 \times 5$ , but  $N = 45$  is not possible since  $45 = 3^2 \times 5$ , so 45 satisfies condition (A), but not condition (B).

The construction of a regular pentagon ( $N = 5$ ) was known by the ancient Greeks. The earliest published construction for the 17-gon was that of Hugenin in 1803. Richelot gave details for the 257-gon in 1832.

Professor Hermes spent 10 years on the 65537-gon, and his work is stored in the University of Göttingen.

### Some things to think about.

- (1) What are the 31 numbers you get by multiplying together any non-empty set of numbers taken from

$$\{3, 5, 17, 257, 65537\}?$$

- (2) List in increasing order, the values of  $N$  with say  $N < 100$  is it possible to draw a regular  $N$ -sided polygon using ruler and compass? If you find 100 easy, try a larger number. The sequence starts

$$3, 4, 5, 6, 8, 10, 12, 15, \dots$$

- (3) One way to construct a regular pentagon using ruler and compasses is to use the fact that

$$\cos(2\pi/5) = \frac{\sqrt{5} - 1}{4}.$$

- How does this fact help?
- Why is this equation true?

- (4) It is a fact that

$$\cos(2\pi/17) = -\frac{1}{16} + \frac{\sqrt{17}}{16} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}$$

Can you use this to draw a regular 17 sided polygon?