

The square root of 2

The square root of 2 is not a fraction, and so its decimal expansion does not repeat. To fifty decimal places, it is

1.41421356237309504880168872420969807856967187537695.

There are various ways of obtaining an accurate approximation to $\sqrt{2}$. Here is a one. For any number x , the product of x and $2/x$ is equal to 2, so they lie on opposite sides of $\sqrt{2}$. Their average

$$x' = \frac{x + 2/x}{2} = \frac{x^2 + 2}{2x}$$

will be closer to $\sqrt{2}$ than x . If repeat this process with x' , and calculate,

$$x'' = \frac{x'^2 + 2}{2x'},$$

and keep repeating, we get a sequence tending to $\sqrt{2}$. For instance, starting with $x = 1$, we get $x' = 3/2$ and then we get the sequence of fractions

1, 3/2, 17/12, 577/408, ...

Calculating these in decimals to 6 decimal places, we get

1.000000, 1.500000, 1.416667, 1.414216 ...

while $\sqrt{2} \approx 1.414214$. The next fraction after $\frac{p}{q}$ is

$$\frac{(p/q)^2 + 2}{2(p/q)} = \frac{p^2 + 2q^2}{2pq}$$

so after 577/408 comes

$$\frac{577^2 + 2 \times 408^2}{2 \times 577 \times 408} = \frac{665857}{470832}.$$

To fifty decimal places, this fraction is

1.41421356237468991062629557889013491011655962211574.

Some things to think about.

- (1) These fractions also appear in the sequence 1, 3/2, 7/5, 17/12, 41/29, Can you work out the next term?
- (2) How would you modify this to approximate $\sqrt{3}$?
- (3) Starting with 1, show that

$$\sqrt{11} \approx \frac{28383073}{8557008}.$$

- (4) How would you modify this to approximate the cube root of 2?